Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise A, Question 1

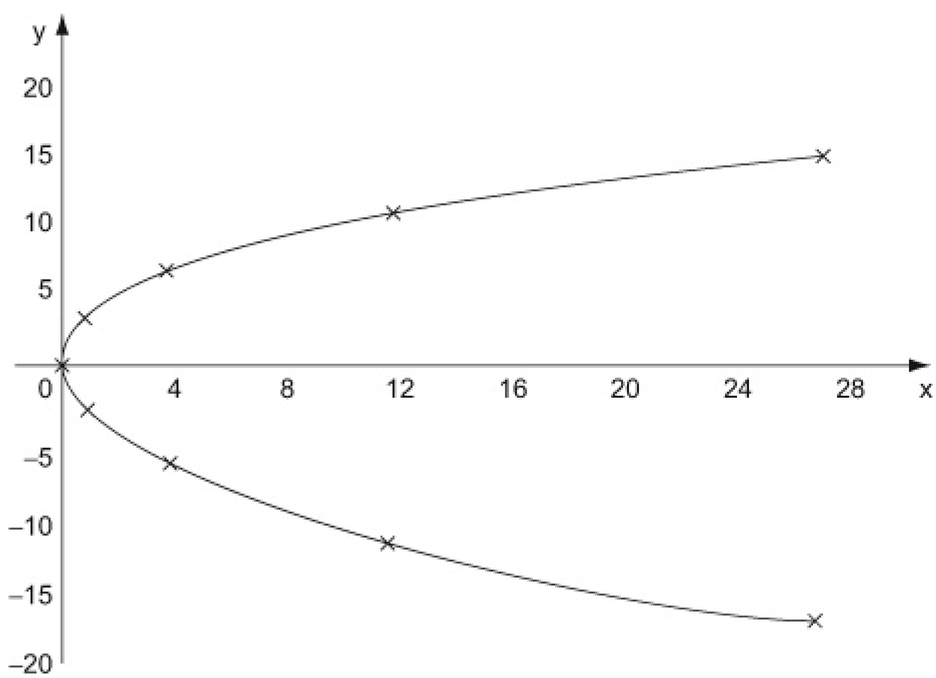
Question:

A curve is given by the parametric equations $x = 2t^2$, $y = 4t$. $t \in \mathbb{R}$. Copy and complete the following table and draw a graph of the curve for $-4 \le t \le 4$.

t	−4	−3	−2	−1	−0.5	0	0.5	1	2	3	4
$x = 2t^2$	32					0	0.5				32
$y = 4t$	−16						2				16

Solution:

t	−4	−3	−2	−1	−0.5	0	0.5	1	2	3	4
$x = 2t^2$	32	18	8	2	0.5	0	0.5	2	8	18	32
$y = 4t$	−16	−12	−8	−4	−2	0	2	4	8	12	16



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Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise A, Question 2

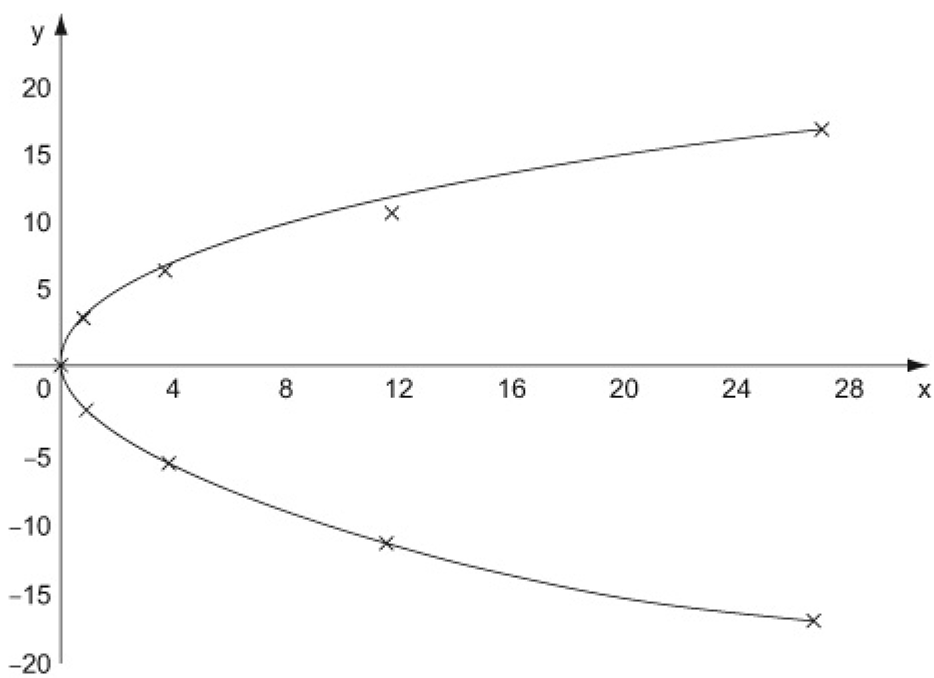
Question:

A curve is given by the parametric equations $x = 3t^2$, $y = 6t$. $t \in \mathbb{R}$. Copy and complete the following table and draw a graph of the curve for $-3 \le t \le 3$.

t	−3	−2	−1	−0.5	0	0.5	1	2	3
$x = 3t^2$					0				
$y = 6t$					0				

Solution:

t	−3	−2	−1	−0.5	0	0.5	1	2	3
$x = 3t^2$	27	12	3	0.75	0	0.75	3	12	27
$y = 6t$	−18	−12	−6	−3	0	3	6	12	18



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Quadratic Equations
Exercise A, Question 3

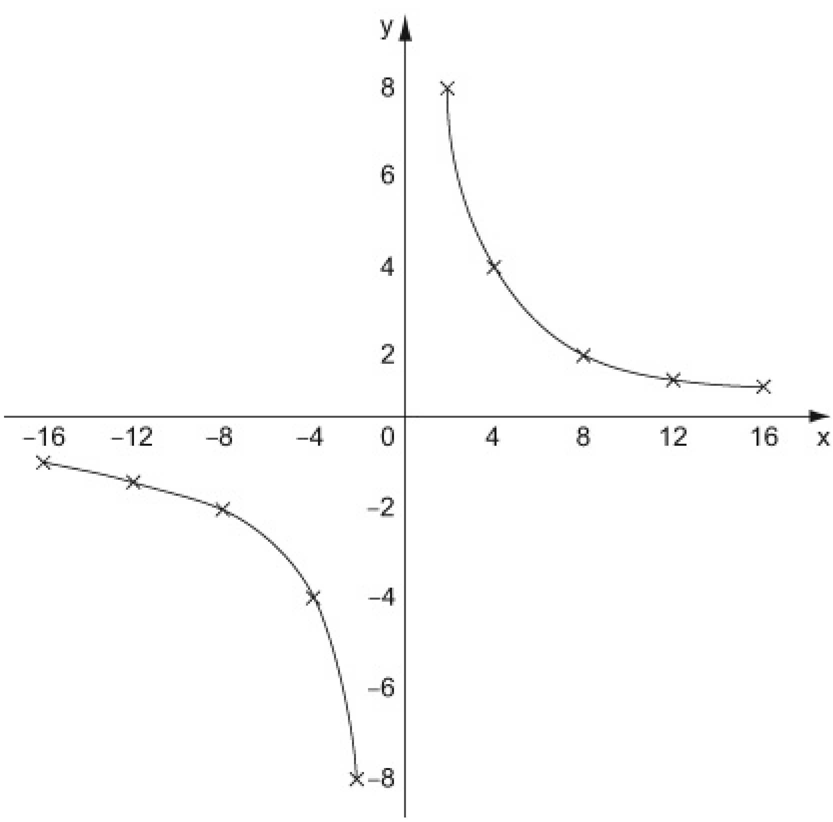
Question:

A curve is given by the parametric equations $x = 4t$, $y = \frac{4}{t}$. $t \in \mathbb{R}$, $t \neq 0$. Copy and complete the following table and draw a graph of the curve for $-4 \leq t \leq 4$.

t	−4	−3	−2	−1	−0.5	0.5	1	2	3	4
$x = 4t$	−16				−2					
$y = \frac{4}{t}$	−1				−8					

Solution:

t	−4	−3	−2	−1	−0.5	0.5	1	2	3	4
$x = 4t$	−16	−12	−8	−4	−2	2	4	8	12	16
$y = \frac{4}{t}$	−1	$-\frac{4}{3}$	−2	−4	−8	8	4	2	$\frac{4}{3}$	1



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Quadratic Equations
Exercise A, Question 4

Question:

Find the Cartesian equation of the curves given by these parametric equations.

a $x = 5t^2,\ \ y = 10t$

b $x = \frac{1}{2}t^2,\ \ y = t$

c $x = 50t^2,\ \ y = 100t$

d $x = \frac{1}{5}t^2,\ \ y = \frac{2}{5}t$

e $x = \frac{5}{2}t^2,\ \ y = 5t$

f $x = \sqrt{3}\,t^2,\ \ y = 2\sqrt{3}\,t$

g $x = 4t,\ \ y = 2t^2$

h $x = 6t,\ \ y = 3t^2$

Solution:

a $y = 10t$

So $t = \frac{y}{10}$ **(1)**

$x = 5t^2$ **(2)**

Substitute **(1)** into **(2):**

$$x = 5\left(\frac{y}{10}\right)^2$$

So $x = \frac{5y^2}{100}$ simplifies to $x = \frac{y^2}{20}$

Hence, the Cartesian equation is $y^2 = 20x$.

b $y = t$ **(1)**

$x = \frac{1}{2}t^2$ **(2)**

Substitute **(1)** into **(2):**

$$x = \frac{1}{2}y^2$$

Hence, the Cartesian equation is $y^2 = 2x$.

c $y = 100t$

So $t = \frac{y}{100}$ **(1)**

$x = 50t^2$ **(2)**

Substitute **(1)** into **(2):**

$$x = 50\left(\frac{y}{100}\right)^2$$

So $x = \frac{50y^2}{10000}$ simplifies to $x = \frac{y^2}{200}$

Hence, the Cartesian equation is $y^2 = 200x$.

d $y = \frac{2}{5}t$

So $t = \frac{5y}{2}$ **(1)**

$x = \frac{1}{5}t^2$ **(2)**

Substitute **(1)** into **(2):**

$$x = \frac{1}{5}\left(\frac{5y}{2}\right)^2$$

So $x = \frac{25y^2}{20}$ simplifies to $x = \frac{5y^2}{4}$

Hence, the Cartesian equation is $y^2 = \frac{4}{5}x$.

e $y = 5t$

So $t = \frac{y}{5}$ **(1)**

$x = \frac{5}{2}t^2$ **(2)**

Substitute **(1)** into **(2):**

$$x = \frac{5}{2}\left(\frac{y}{5}\right)^2$$

So $x = \frac{5y^2}{50}$ simplifies to $x = \frac{y^2}{10}$

Hence, the Cartesian equation is $y^2 = 10x$.

f $y = 2\sqrt{3}t$

So $t = \frac{y}{2\sqrt{3}}$ **(1)**

$x = \sqrt{3}t^2$ **(2)**

Substitute **(1)** into **(2):**

$$x = \sqrt{3}\left(\frac{y}{2\sqrt{3}}\right)^2$$

So $x = \frac{\sqrt{3}y^2}{12}$ gives $y = \frac{12x}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

Hence, the Cartesian equation is $y^2 = 4\sqrt{3}x$.

g $x = 4t$

So $t = \frac{x}{4}$ **(1)**

$y = 2t^2$ **(2)**

Substitute **(1)** into **(2):**

$$y = 2\left(\frac{x}{4}\right)^2$$

So $y = \frac{2x^2}{16}$ simplifies to $y = \frac{x^2}{8}$

Hence, the Cartesian equation is $x^2 = 8y$.

h $x = 6t$

So $t = \frac{x}{6}$ **(1)**

$y = 3t^2$ **(2)**

Substitute **(1)** into **(2):**

$$y = 3\left(\frac{x}{6}\right)^2$$

So $y = \frac{3x^2}{36}$ simplifies to $y = \frac{x^2}{12}$

Hence, the Cartesian equation is $x^2 = 12y$.

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Quadratic Equations
Exercise A, Question 5

Question:

Find the Cartesian equation of the curves given by these parametric equations.

a $x = t,\ \ y = \frac{1}{t},\ \ t \neq 0$

b $x = 7t,\ \ y = \frac{7}{t},\ \ t \neq 0$

c $x = 3\sqrt{5}t,\ \ y = \frac{3\sqrt{5}}{t},\ \ t \neq 0$

d $x = \frac{t}{5},\ \ y = \frac{1}{5t},\ \ t \neq 0$

Solution:

a $xy = t \times \left(\frac{1}{t}\right)$

$xy = \frac{t}{t}$

Hence, the Cartesian equation is $xy = 1$.

b $xy = 7t \times \left(\frac{7}{t}\right)$

$xy = \frac{49t}{t}$

Hence, the Cartesian equation is $xy = 49$.

c $xy = 3\sqrt{5}t \times \left(\frac{3\sqrt{5}}{t}\right)$

$xy = \frac{9(5)t}{t}$

Hence, the Cartesian equation is $xy = 45$.

d $xy = \frac{t}{5} \times \left(\frac{1}{5t}\right)$

$xy = \frac{t}{25t}$

Hence, the Cartesian equation is $xy = \frac{1}{25}$.

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Quadratic Equations
Exercise A, Question 6

Question:

A curve has parametric equations $x = 3t,\ \ y = \frac{3}{t},\ \ t \in \mathbb{R}, t \neq 0.$

a Find the Cartesian equation of the curve.

b Hence sketch this curve.

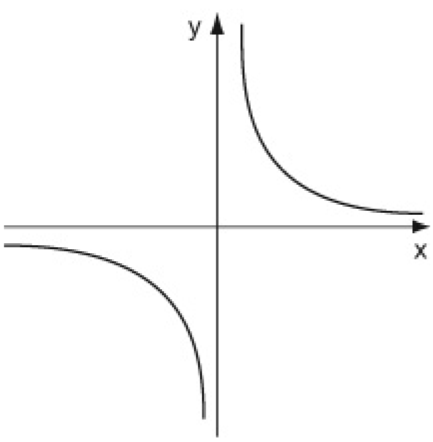
Solution:

a $xy = 3t \times \left(\frac{3}{t}\right)$

$xy = \frac{9t}{t}$

Hence, the Cartesian equation is $xy = 9$.

b



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Quadratic Equations
Exercise A, Question 7

Question:

A curve has parametric equations $x = \sqrt{2}t,\ \ y = \frac{\sqrt{2}}{t},\ \ t \in \mathbb{R}$, $t \neq 0$.

a Find the Cartesian equation of the curve.

b Hence sketch this curve.

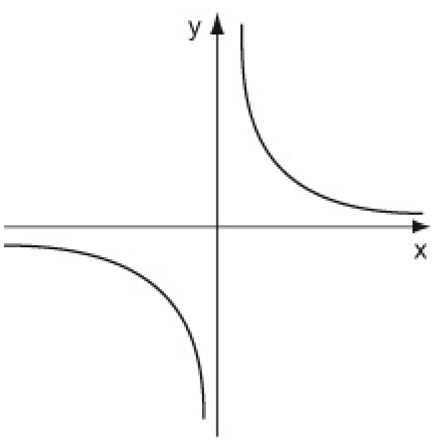
Solution:

a $xy = \sqrt{2}t \times \left(\frac{\sqrt{2}}{t}\right)$

$xy = \frac{2t}{t}$

Hence, the Cartesian equation is $xy = 2$.

b



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Quadratic Equations
Exercise B, Question 1

Question:

Find an equation of the parabola with

a focus (5, 0) and directrix $x+5=0$,

b focus (8, 0) and directrix $x+8=0$,

c focus (1, 0) and directrix $x=-1$,

d focus $\left(\frac{3}{2},\ 0\right)$ and directrix $x=-\frac{3}{2}$,

e focus $\left(\frac{\sqrt{3}}{2},\ 0\right)$ and directrix $x+\frac{\sqrt{3}}{2}=0$.

Solution:

The focus and directrix of a parabola with equation $y^2=4ax$, are $(a, 0)$ and $x+a=0$ respectively.

a focus (5, 0) and directrix $x+5=0$.

So $a=5$ and $y^2=4(5)x$.

Hence parabola has equation $y^2=20x$.

b focus (8, 0) and directrix $x+8=0$.

So $a=8$ and $y^2=4(8)x$.

Hence parabola has equation $y^2=32x$.

c focus (1, 0) and directrix $x=-1$ giving $x+1=0$.

So $a=1$ and $y^2=4(1)x$.

Hence parabola has equation $y^2=4x$.

d focus $\left(\frac{3}{2}, 0\right)$ and directrix $x=-\frac{3}{2}$ giving $x+\frac{3}{2}=0$.

So $a=\frac{3}{2}$ and $y^2=4\left(\frac{3}{2}\right)x$.

Hence parabola has equation $y^2=6x$.

e focus $\left(\frac{\sqrt{3}}{2}, 0\right)$ and directrix $x+\frac{\sqrt{3}}{2}=0$.

So $a=\frac{\sqrt{3}}{2}$ and $y^2=4\left(\frac{\sqrt{3}}{2}\right)x$.

Hence parabola has equation $y^2 = 2\sqrt{3}x$.

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Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise B, Question 2

Question:

Find the coordinates of the focus, and an equation for the directrix of a parabola with these equations.

a $y^2 = 12x$

b $y^2 = 20x$

c $y^2 = 10x$

d $y^2 = 4\sqrt{3}x$

e $y^2 = \sqrt{2}x$

f $y^2 = 5\sqrt{2}x$

Solution:

The focus and directrix of a parabola with equation $y^2 = 4ax$, are $(a, 0)$ and $x + a = 0$ respectively.

a $y^2 = 12x$. So $4a = 12$, gives $a = \frac{12}{4} = 3$.

So the focus has coordinates (3, 0) and the directrix has equation $x + 3 = 0$.

b $y^2 = 20x$. So $4a = 20$, gives $a = \frac{20}{4} = 5$.

So the focus has coordinates (5, 0) and the directrix has equation $x + 5 = 0$.

c $y^2 = 10x$. So $4a = 10$, gives $a = \frac{10}{4} = \frac{5}{2}$.

So the focus has coordinates $\left(\frac{5}{2}, 0\right)$ and the directrix has equation $x + \frac{5}{2} = 0$.

d $y^2 = 4\sqrt{3}x$. So $4a = 4\sqrt{3}$, gives $a = \frac{4\sqrt{3}}{4} = \sqrt{3}$.

So the focus has coordinates $(\sqrt{3}, 0)$ and the directrix has equation $x + \sqrt{3} = 0$.

e $y^2 = \sqrt{2}x$. So $4a = \sqrt{2}$, gives $a = \frac{\sqrt{2}}{4}$.

So the focus has coordinates $\left(\frac{\sqrt{2}}{4}, 0\right)$ and the directrix has equation $x + \frac{\sqrt{2}}{4} = 0$.

f $y^2 = 5\sqrt{2}x$. So $4a = 5\sqrt{2}$, gives $a = \frac{5\sqrt{2}}{4}$.

So the focus has coordinates $\left(\frac{5\sqrt{2}}{4}, 0\right)$ and the directrix has equation $x + \frac{5\sqrt{2}}{4} = 0$.

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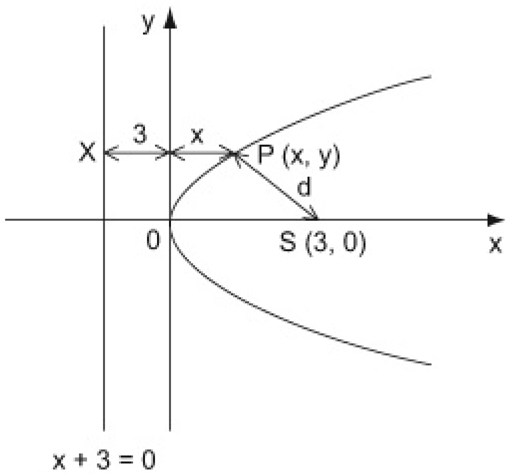
Quadratic Equations
Exercise B, Question 3

Question:

A point $P(x, y)$ obeys a rule such that the distance of P to the point (3, 0) is the same as the distance of P to the straight line $x + 3 = 0$. Prove that the locus of P has an equation of the form $y^2 = 4ax$, stating the value of the constant a.

Solution:



The (shortest) distance of P to the line $x + 3 = 0$ is the distance XP.

The distance SP is the same as the distance XP.

The line XP is horizontal and has distance $XP = x + 3$.

The locus of P is the curve shown.

From sketch the locus satisfies $SP = XP$.

This means the distance SP is the same as the distance XP.

Therefore, $SP^2 = XP^2$.
So, $(x-3)^2 + (y-0)^2 = (x--3)^2$.

Use $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ on $SP^2 = XP^2$, where $S(3, 0)$, $P(x, y)$, and $X(-3, y)$.

$$x^2 - 6x + 9 + y^2 = x^2 + 6x + 9$$
$$-6x + y^2 = 6x$$

which simplifies to $y^2 = 12x$.

This is in the form $y^2 = 4ax$.
So $4a = 12$, gives $a = \frac{12}{4} = 3$.

So, the locus of P has an equation of the form $y^2 = 4ax$, where $a = 3$.

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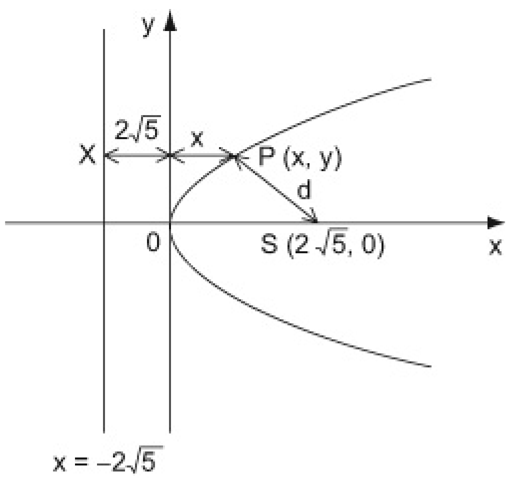
Quadratic Equations
Exercise B, Question 4

Question:

A point $P(x, y)$ obeys a rule such that the distance of P to the point $(2\sqrt{5},\ 0)$ is the same as the distance of P to the straight line $x = -2\sqrt{5}$. Prove that the locus of P has an equation of the form $y^2 = 4ax$, stating the value of the constant a.

Solution:



The (shortest) distance of P to the line $x = -2\sqrt{5}$ or $x + 2\sqrt{5} = 0$ is the distance XP.

The distance SP is the same as the distance XP.

The line XP is horizontal and has distance $XP = x + 2\sqrt{5}$.

The locus of P is the curve shown.

From sketch the locus satisfies $SP = XP$.

This means the distance SP is the same as the distance XP.

Therefore, $SP^2 = XP^2$.

So, $(x - 2\sqrt{5})^2 + (y - 0)^2 = (x - -2\sqrt{5})^2$.

Use $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ on $SP^2 = XP^2$, where $S(2\sqrt{5}, 0)$, $P(x, y)$, and $X(-2\sqrt{5}, y)$.

$$x^2 - 4\sqrt{5}x + 20 + y^2 = x^2 + 4\sqrt{5}x + 20$$
$$-4\sqrt{5}x + y^2 = 4\sqrt{5}x$$

which simplifies to $y^2 = 8\sqrt{5}x$.

This is in the form $y^2 = 4ax$.

So $4a = 8\sqrt{5}$, gives $a = \frac{8\sqrt{5}}{4} = 2\sqrt{5}$.

So, the locus of P has an equation of the form $y^2 = 4ax$, where $a = 2\sqrt{5}$.

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Quadratic Equations
Exercise B, Question 5

Question:

A point $P(x, y)$ obeys a rule such that the distance of P to the point (0, 2) is the same as the distance of P to the straight line $y = -2$.

a Prove that the locus of P has an equation of the form $y = kx^2$, stating the value of the constant k.

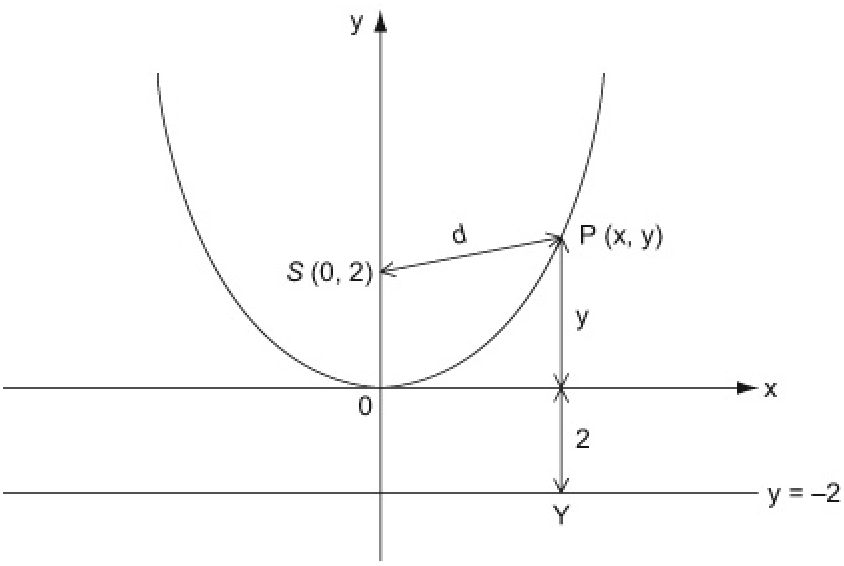
Given that the locus of P is a parabola,

b state the coordinates of the focus of P, and an equation of the directrix to P,

c sketch the locus of P with its focus and its directrix.

Solution:

a



The (shortest) distance of P to the line $y = -2$ is the distance YP.

The distance SP is the same as the distance YP.

The line YP is vertical and has distance $YP = y + 2$.

The locus of P is the curve shown.

From sketch the locus satisfies $SP = YP$.

This means the distance SP is the same as the distance YP.

Therefore, $SP^2 = YP^2$.

So, $(x-0)^2 + (y-2)^2 = (y--2)^2$.

Use $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ on $SP^2 = YP^2$, where $S(0, 2)$, $P(x, y)$, and $Y(x, -2)$.

$$x^2 + y^2 - 4y + 4 = y^2 + 4y + 4$$
$$x^2 - 4y = 4y$$

which simplifies to $x^2 = 8y$ and then $y = \frac{1}{8}x^2$.

So, the locus of P has an equation of the form $y = \frac{1}{8}x^2$, where

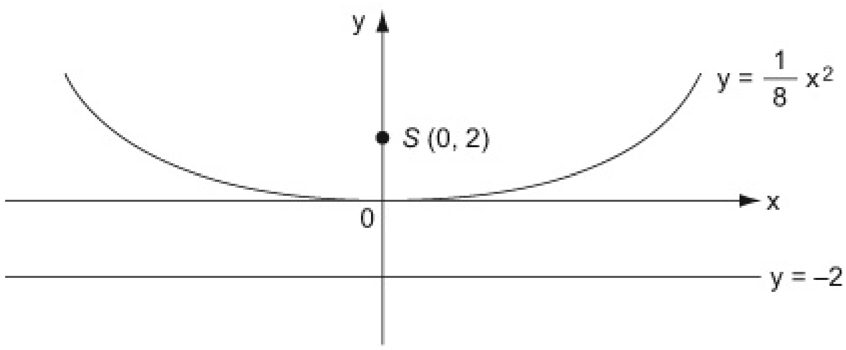
$k = \frac{1}{8}$.

b The focus and directrix of a parabola with equation $y^2 = 4ax$, are $(a, 0)$ and $x + a = 0$ respectively. Therefore it follows that the focus and directrix of a parabola with equation $x^2 = 4ay$, are $(0, a)$ and $y + a = 0$ respectively.

So the focus has coordinates (0, 2) and the directrix has equation $y + 2 = 0$.

$x^2 = 8y$ is in the form $x^2 = 4ay$.
So $4a = 8$, gives $a = \frac{8}{4} = 2$.

c



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Quadratic Equations
Exercise C, Question 1

Question:

The line $y = 2x - 3$ meets the parabola $y^2 = 3x$ at the points P and Q.

Find the coordinates of P and Q.

Solution:

Line: $y = 2x - 3$ **(1)**

Curve: $y^2 = 3x$ **(2)**

Substituting **(1)** into **(2)** gives

$$(2x - 3)^2 = 3x$$
$$(2x - 3)(2x - 3) = 3x$$
$$4x^2 - 12x + 9 = 3x$$
$$4x^2 - 15x + 9 = 0$$
$$(x - 3)(4x - 3) = 0$$
$$x = 3, \frac{3}{4}$$

When $x = 3$, $y = 2(3) - 3 = 3$

When $x = \frac{3}{4}$, $y = 2\left(\frac{3}{4}\right) - 3 = -\frac{3}{2}$

Hence the coordinates of P and Q are (3, 3) and $\left(\frac{3}{4}, -\frac{3}{2}\right)$.

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Quadratic Equations
Exercise C, Question 2

Question:

The line $y = x + 6$ meets the parabola $y^2 = 32x$ at the points *A* and *B*. Find the exact length *AB* giving your answer as a surd in its simplest form.

Solution:

Line: $y = x + 6$ **(1)**

Curve: $y^2 = 32x$ **(2)**

Substituting **(1)** into **(2)** gives

$$\begin{aligned}
&(x+6)^2 = 32x \\
&(x+6)(x+6) = 32x \\
&x^2 + 12x + 36 = 32x \\
&x^2 - 20x + 36 = 0 \\
&(x-2)(x-18) = 0 \\
&x = 2, 18
\end{aligned}$$

When $x = 2$, $y = 2 + 6 = 8$.

When $x = 18$, $y = 18 + 6 = 24$.

Hence the coordinates of *A* and *B* are (2, 8) and (18, 24).

$$\begin{aligned}
AB &= \sqrt{(18-2)^2 + (24-8)^2} \quad \text{Use } d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \\
&= \sqrt{16^2 + 16^2} \\
&= \sqrt{2(16)^2} \\
&= 16\sqrt{2}
\end{aligned}$$

Hence the exact length *AB* is $16\sqrt{2}$.

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Quadratic Equations
Exercise C, Question 3

Question:

The line $y = x - 20$ meets the parabola $y^2 = 10x$ at the points A and B. Find the coordinates of A and B. The mid-point of AB is the point M. Find the coordinates of M.

Solution:

Line: $y = x - 20$ **(1)**

Curve: $y^2 = 10x$ **(2)**

Substituting **(1)** into **(2)** gives

$$(x-20)^2 = 10x$$
$$(x-20)(x-20) = 10x$$
$$x^2 - 40x + 400 = 10x$$
$$x^2 - 50x + 400 = 0$$
$$(x-10)(x-40) = 0$$
$$x = 10, 40$$

When $x = 10,\ y = 10 - 20 = -10.$

When $x = 40,\ y = 40 - 20 = 20.$

Hence the coordinates of A and B are (10, −10) and (40, 20).

The midpoint of A and B is $\left(\frac{10+40}{2}, \frac{-10+20}{2}\right) = (25, 5)$. Use $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Hence the coordinates of M are (25, 5).

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Quadratic Equations
Exercise C, Question 4

Question:

The parabola C has parametric equations $x = 6t^2$, $y = 12t$. The focus to C is at the point S.

a Find a Cartesian equation of C.

b State the coordinates of S and the equation of the directrix to C.

c Sketch the graph of C.

The points P and Q are both at a distance 9 units away from the directrix of the parabola.

d State the distance PS.

e Find the exact length PQ, giving your answer as a surd in its simplest form.

f Find the area of the triangle PQS, giving your answer in the form $k\sqrt{2}$, where k is an integer.

Solution:

a $y = 12t$

So $t = \frac{y}{12}$ **(1)**

$x = 6t^2$ **(2)**

Substitute **(1)** into **(2)**:

$$x = 6\left(\frac{y}{12}\right)^2$$

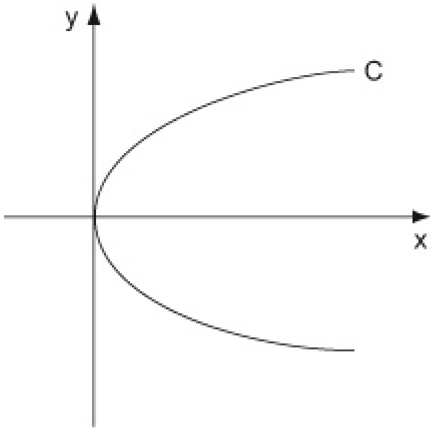
So $x = \frac{6y^2}{144}$ simplifies to $x = \frac{y^2}{24}$

Hence, the Cartesian equation is $y^2 = 24x$.

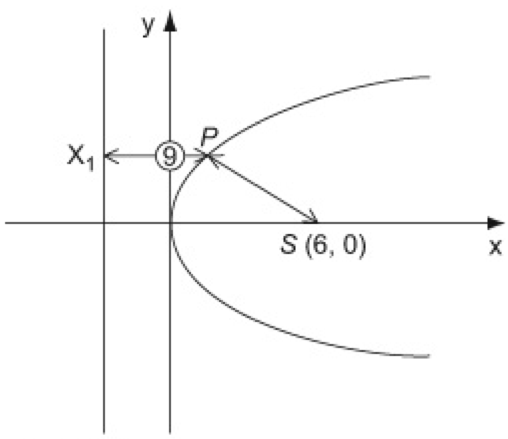
b $y^2 = 24x$. So $4a = 24$, gives $a = \frac{24}{4} = 6$.

So the focus S, has coordinates (6, 0) and the directrix has equation $x + 6 = 0$.

c



d



The (shortest) distance of P to the line $x + 6 = 0$ is the distance X_1P.

Therefore $X_1P = 9$.

The distance PS is the same as the distance X_1P, by the focus-directrix property.

Hence the distance $PS = 9$.

e Using diagram in (d), the x-coordinate of P and Q is $x = 9 - 6 = 3$.

When $x = 3$, $y^2 = 24(3) = 72$.

$$\begin{aligned} \text{Hence } y &= \pm\sqrt{72} \\ &= \pm\sqrt{36}\sqrt{2} \\ &= \pm 6\sqrt{2} \end{aligned}$$

So the coordinates are of P and Q are $(3, 6\sqrt{2})$ and $(3, -6\sqrt{2})$.

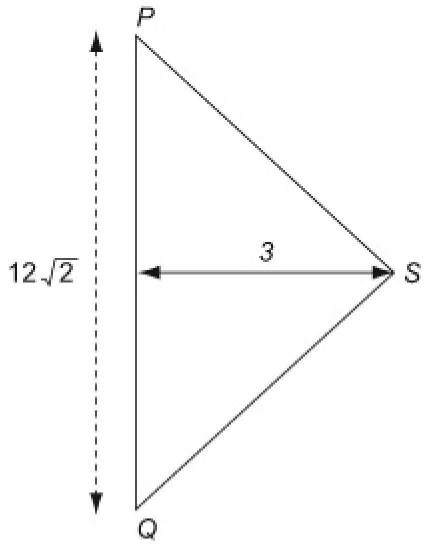
As P and Q are vertically above each other then

$$\begin{aligned} PQ &= 6\sqrt{2} - -6\sqrt{2} \\ &= 12\sqrt{2}. \end{aligned}$$

Hence, the distance PQ is $12\sqrt{2}$.

f Drawing a diagram of the triangle PQS gives:

The x-coordinate of P and Q is 3 and the x-coordinate of S is 6.



Hence the height of the triangle is height $= 6 - 3 = 3$.

The length of the base is $12\sqrt{2}$.

$$\text{Area} = \frac{1}{2}(12\sqrt{2})(3)$$
$$= \frac{1}{2}(36\sqrt{2})$$
$$= 18\sqrt{2}.$$

Therefore the area of the triangle is $18\sqrt{2}$, where $k = 18$.

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Quadratic Equations
Exercise C, Question 5

Question:

The parabola C has equation $y^2 = 4ax$, where a is a constant. The point $\left(\frac{5}{4}t^2, \ \frac{5}{2}t\right)$ is a general point on C.

a Find a Cartesian equation of C.

The point P lies on C with y-coordinate 5.

b Find the x-coordinate of P.

The point Q lies on the directrix of C where $y = 3$. The line l passes through the points P and Q.

c Find the coordinates of Q.

d Find an equation for l, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

Solution:

a $P\left(\frac{5}{4}t^2, \frac{5}{2}t\right)$. Substituting $x = \frac{5}{4}t^2$ and $y = \frac{5}{2}t$ into $y^2 = 4ax$ gives,

$$\left(\frac{5}{2}t\right)^2 = 4a\left(\frac{5}{4}t^2\right) \Rightarrow \frac{25t^2}{4} = 5at^2 \Rightarrow \frac{25}{4} = 5a \Rightarrow \frac{5}{4} = a$$

When $a = \frac{5}{4}, y^2 = 4\left(\frac{5}{4}\right)x \Rightarrow y^2 = 5x$

The Cartesian equation of C is $y^2 = 5x$.

b When $y = 5, (5)^2 = 5x \Rightarrow \frac{25}{5} = x \Rightarrow x = 5.$

The x-coordinate of P is 5.

c As $a = \frac{5}{4}$, the equation of the directrix of C is $x + \frac{5}{4} = 0$ or $x = -\frac{5}{4}$.

Therefore the coordinates of Q are $\left(-\frac{5}{4}, 3\right)$.

d The coordinates of P and Q are (5, 5) and $\left(-\frac{5}{4}, 3\right)$.

$$m_l = m_{PQ} = \frac{3-5}{-\frac{5}{4}-5} = \frac{-2}{-\frac{25}{4}} = \frac{8}{25}$$

$$l : y - 5 = \frac{8}{25}(x-5)$$
$$l : 25y - 125 = 8(x-5)$$
$$l : 25y - 125 = 8x - 40$$
$$l : 0 = 8x - 25y - 40 + 125$$
$$l : 0 = 8x - 25y + 85$$

An equation for l is $8x - 25y + 85 = 0$.

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Quadratic Equations
Exercise C, Question 6

Question:

A parabola C has equation $y^2 = 4x$. The point S is the focus to C.

a Find the coordinates of S.

The point P with y-coordinate 4 lies on C.

b Find the x-coordinate of P.

The line l passes through S and P.

c Find an equation for l, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

The line l meets C again at the point Q.

d Find the coordinates of Q.

e Find the distance of the directrix of C to the point Q.

Solution:

a $y^2 = 4x$. So $4a = 4$, gives $a = \frac{4}{4} = 1$.

So the focus S, has coordinates $(1, 0)$.

Also note that the directrix has equation $x + 1 = 0$.

b Substituting $y = 4$ into $y^2 = 4x$ gives:

$$16 = 4x \Rightarrow x = \frac{16}{4} = 4.$$

The x-coordinate of P is 4.

c The line l goes through $S(1, 0)$ and $P(4, 4)$.

Hence gradient of l, $m_l = \frac{4-0}{4-1} = \frac{4}{3}$

$$\begin{aligned} \text{Hence, } y - 0 &= \frac{4}{3}(x-1) \\ 3y &= 4(x-1) \\ 3y &= 4x - 4 \\ 0 &= 4x - 3y - 4 \end{aligned}$$

The line l has equation $4x - 3y - 4 = 0$.

d Line $l : 4x - 3y - 4 = 0$ **(1)**

Curve : $y^2 = 4x$ **(2)**

Substituting **(2)** into **(1)** gives

$y^2 - 3y - 4 = 0$
$(y-4)(y+1) = 0$
$y = 4, -1$

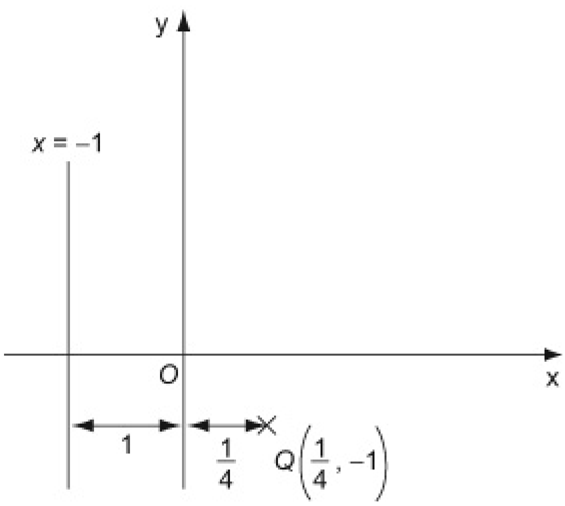
At P, it is already known that $y = 4$. So at Q, $y = -1$.

Substituting $y = -1$ into $y^2 = 4x$ gives

$(-1)^2 = 4x \Rightarrow x = \frac{1}{4}$.

Hence the coordinates of Q are $\left(\frac{1}{4}, -1\right)$.

e The directrix of C has equation $x + 1 = 0$ or $x = -1$. Q has coordinates $\left(\frac{1}{4}, -1\right)$.



From the diagram, distance $= 1 + \frac{1}{4} = \frac{5}{4}$.

Therefore the distance of the directrix of C to the point Q is $\frac{5}{4}$.

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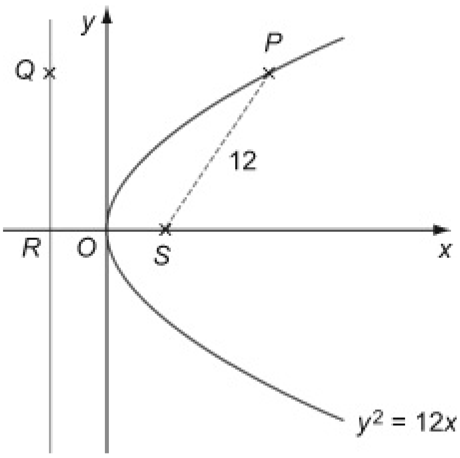
Quadratic Equations
Exercise C, Question 7

Question:

The diagram shows the point P which lies on the parabola C with equation $y^2 = 12x$.



The point S is the focus of C. The points Q and R lie on the directrix to C. The line segment QP is parallel to the line segment RS as shown in the diagram. The distance of PS is 12 units.

a Find the coordinates of R and S.

b Hence find the exact coordinates of P and Q.

c Find the area of the quadrilateral $PQRS$, giving your answer in the form $k\sqrt{3}$, where k is an integer.

Solution:

a $y^2 = 12x$. So $4a = 12$, gives $a = \frac{12}{4} = 3$.

Therefore the focus S has coordinates (3, 0) and an equation of the directrix of C is $x + 3 = 0$ or $x = -3$. The coordinates of R are (-3, 0) as R lies on the x-axis.

b The directrix has equation $x = -3$. The (shortest) distance of P to the directrix is the distance PQ. The distance $SP = 12$. The focus-directrix property implies that $SP = PQ = 12$.

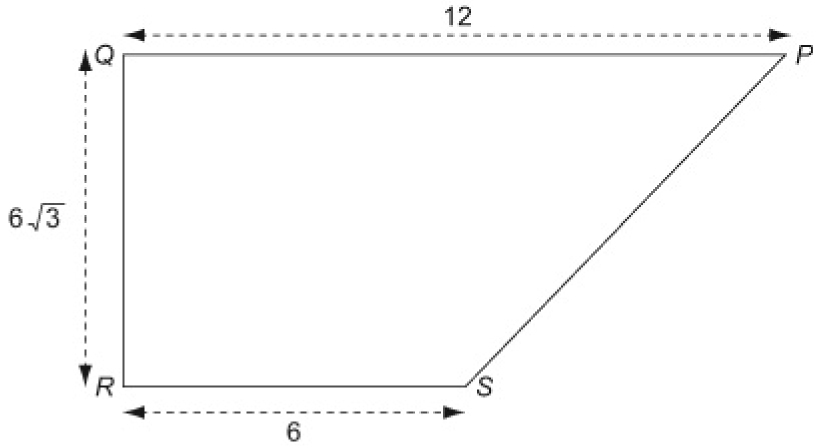
Therefore the x-coordinate of P is $x = 12 - 3 = 9$.

As P lies on C, when $x = 9, y^2 = 12(9) \Rightarrow y^2 = 108$

As $y > 0$, $y = \sqrt{108} = \sqrt{36}\sqrt{3} = 6\sqrt{3} \Rightarrow P(9, 6\sqrt{3})$

Hence the exact coordinates of P are $(9, 6\sqrt{3})$ and the coordinates of Q are $(-3, 6\sqrt{3})$.

c



$$\begin{aligned}\text{Area}(PQRS) &= \frac{1}{2}(6+12)6\sqrt{3}\\ &= \frac{1}{2}(18)(6\sqrt{3})\\ &= (9)(6\sqrt{3})\\ &= 54\sqrt{3}\end{aligned}$$

The area of the quadrilateral *PQRS* is $54\sqrt{3}$ and $k = 54$.

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Quadratic Equations
Exercise C, Question 8

Question:

The points $P(16, 8)$ and $Q(4, b)$, where $b < 0$ lie on the parabola C with equation $y^2 = 4ax$.

a Find the values of a and b.

P and Q also lie on the line l. The mid-point of PQ is the point R.

b Find an equation of l, giving your answer in the form $y = mx + c$, where m and c are constants to be determined.

c Find the coordinates of R.

The line n is perpendicular to l and passes through R.

d Find an equation of n, giving your answer in the form $y = mx + c$, where m and c are constants to be determined.

The line n meets the parabola C at two points.

e Show that the x-coordinates of these two points can be written in the form $x = \lambda \pm \mu\sqrt{13}$, where λ and μ are integers to be determined.

Solution:

a $P(16, 8)$. Substituting $x = 16$ and $y = 8$ into $y^2 = 4ax$ gives,

$(8)^2 = 4a(16) \Rightarrow 64 = 64a \Rightarrow a = \frac{64}{64} = 1.$

$Q(4, b)$. Substituting $x = 4$, $y = b$ and $a = 1$ into $y^2 = 4ax$ gives,

$b^2 = 4(1)(4) = 16 \Rightarrow b = \pm\sqrt{16} \Rightarrow b = \pm 4$. As $b < 0$, $b = -4$.

Hence, $a = 1, b = -4$.

b The coordinates of P and Q are $(16, 8)$ and $(4, -4)$.

$m_l = m_{PQ} = \frac{-4-8}{4-16} = \frac{-12}{-12} = 1$

$l : y - 8 = 1(x - 16)$

$l : y = x - 8$

l has equation $y = x - 8$.

c R has coordinates $\left(\frac{16+4}{2}, \frac{8+-4}{2}\right) = (10, 2)$.

d As n is perpendicular to l, $m_n = -1$

$n : y - 2 = -1(x - 10)$

$n : y - 2 = -x + 10$

$n : y = -x + 12$

n has equation $y = -x + 12$.

e Line n : $y = -x + 12$ **(1)**

Parabola C : $y^2 = 4x$ **(2)**

Substituting **(1)** into **(2)** gives

$$(-x + 12)^2 = 4x$$
$$x^2 - 12x - 12x + 144 = 4x$$
$$x^2 - 28x + 144 = 0$$
$$(x - 14)^2 - 196 + 144 = 0$$
$$(x - 14)^2 - 52 = 0$$
$$(x - 14)^2 = 52$$
$$x - 14 = \pm\sqrt{52}$$
$$x - 14 = \pm\sqrt{4}\sqrt{13}$$
$$x - 14 = \pm 2\sqrt{13}$$
$$x = 14 \pm 2\sqrt{13}$$

The x coordinates are $x = 14 \pm 2\sqrt{13}$.

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Quadratic Equations
Exercise D, Question 1

Question:

Find the equation of the tangent to the curve

a $y^2 = 4x$ at the point (16, 8)

b $y^2 = 8x$ at the point $(4,\ 4\sqrt{2})$

c $xy = 25$ at the point (5, 5)

d $xy = 4$ at the point where $x = \frac{1}{2}$

e $y^2 = 7x$ at the point $(7,\ -7)$

f $xy = 16$ at the point where $x = 2\sqrt{2}$.

Give your answers in the form $ax + by + c = 0$.

Solution:

a As $y > 0$ in the coordinates (16, 8), then

$$y^2 = 4x \Rightarrow y = \sqrt{4x} = \sqrt{4}\sqrt{x} = 2x^{\frac{1}{2}}$$

So $y = 2x^{\frac{1}{2}}$

$$\frac{dy}{dx} = 2\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = x^{-\frac{1}{2}}$$

So, $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$

At (16, 8), $m_T = \frac{dy}{dx} = \frac{1}{\sqrt{16}} = \frac{1}{4}$.

T: $y - 8 = \frac{1}{4}(x - 16)$

T: $4y - 32 = x - 16$

T: $0 = x - 4y - 16 + 32$

T: $x - 4y + 16 = 0$

Therefore, the equation of the tangent is $x - 4y + 16 = 0$.

b As $y > 0$ in the coordinates $(4, 4\sqrt{2})$, then

$$y^2 = 8x \Rightarrow y = \sqrt{8x} = \sqrt{8}\sqrt{x} = \sqrt{4}\sqrt{2}\sqrt{x} = 2\sqrt{2}\,x^{\frac{1}{2}}$$

So $y = 2\sqrt{2}x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2\sqrt{2}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \sqrt{2}x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = \frac{\sqrt{2}}{\sqrt{x}}$

At $(4, 4\sqrt{2})$, $m_T = \frac{dy}{dx} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}$.

T: $y - 4\sqrt{2} = \frac{\sqrt{2}}{2}(x-4)$

T: $2y - 8\sqrt{2} = \sqrt{2}(x-4)$

T: $2y - 8\sqrt{2} = \sqrt{2}x - 4\sqrt{2}$

T: $0 = \sqrt{2}x - 2y - 4\sqrt{2} + 8\sqrt{2}$

T: $\sqrt{2}x - 2y + 4\sqrt{2} = 0$

Therefore, the equation of the tangent is $\sqrt{2}x - 2y + 4\sqrt{2} = 0$.

c $xy = 25 \Rightarrow y = 25x^{-1}$

$\frac{dy}{dx} = -25x^{-2} = -\frac{25}{x^2}$

At (5, 5), $m_T = \frac{dy}{dx} = -\frac{25}{5^2} = -\frac{25}{25} = -1$

T: $y - 5 = -1(x-5)$

T: $y - 5 = -x + 5$

T: $x + y - 5 - 5 = 0$

T: $x + y - 10 = 0$

Therefore, the equation of the tangent is $x + y - 10 = 0$.

d $xy = 4 \Rightarrow y = 4x^{-1}$

$\frac{dy}{dx} = -4x^{-2} = -\frac{4}{x^2}$

At $x = \frac{1}{2}$, $m_T = \frac{dy}{dx} = -\frac{4}{\left(\frac{1}{2}\right)^2} = -\frac{4}{\left(\frac{1}{4}\right)} = -16$

When $x = \frac{1}{2}$, $y = \frac{4}{\left(\frac{1}{2}\right)} = 8 \Rightarrow \left(\frac{1}{2}, 8\right)$

T: $y - 8 = -16\left(x - \frac{1}{2}\right)$

T: $y - 8 = -16x + 8$

T: $16x + y - 8 - 8 = 0$

T: $16x + y - 16 = 0$

Therefore, the equation of the tangent is $16x + y - 16 = 0$.

e As $y < 0$ in the coordinates $(7, -7)$, then

$y^2 = 7x \Rightarrow y = -\sqrt{7x} = -\sqrt{7}\sqrt{x} = -\sqrt{7}x^{\frac{1}{2}}$

So $y = -\sqrt{7}x^{\frac{1}{2}}$

$\frac{dy}{dx} = -\sqrt{7}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = -\frac{\sqrt{7}}{2}x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = -\frac{\sqrt{7}}{2\sqrt{x}}$

At $(7, -7)$, $m_T = \frac{dy}{dx} = -\frac{\sqrt{7}}{2\sqrt{7}} = -\frac{1}{2}$.

T: $y + 7 = -\frac{1}{2}(x - 7)$

T: $2y + 14 = -1(x - 7)$

T: $2y + 14 = -x + 7$

T: $x + 2y + 14 - 7 = 0$

T: $x + 2y + 7 = 0$

Therefore, the equation of the tangent is $x + 2y + 7 = 0$.

f $xy = 16 \Rightarrow y = 16x^{-1}$

$\frac{dy}{dx} = -16x^{-2} = -\frac{16}{x^2}$

At $x = 2\sqrt{2}$, $m_T = \frac{dy}{dx} = -\frac{16}{(2\sqrt{2})^2} = -\frac{16}{8} = -2$

When $x = 2\sqrt{2}$, $y = \frac{16}{2\sqrt{2}} = \frac{8}{\sqrt{2}} = \frac{8\sqrt{2}}{\sqrt{2}\sqrt{2}} = 4\sqrt{2} \Rightarrow (2\sqrt{2}, 4\sqrt{2})$

T: $y - 4\sqrt{2} = -2(x - 2\sqrt{2})$

T: $y - 4\sqrt{2} = -2x + 4\sqrt{2}$

T: $2x + y - 4\sqrt{2} - 4\sqrt{2} = 0$

T: $2x + y - 8\sqrt{2} = 0$

Therefore, the equation of the tangent is $2x + y - 8\sqrt{2} = 0$.

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Quadratic Equations
Exercise D, Question 2

Question:

Find the equation of the normal to the curve

a $y^2 = 20x$ at the point where $y = 10$,

b $xy = 9$ at the point $\left(-\frac{3}{2},\ -6\right)$.

Give your answers in the form $ax + by + c = 0$, where a, b and c are integers.

Solution:

a Substituting $y = 10$ into $y^2 = 20x$ gives

$(10)^2 = 20x \Rightarrow x = \frac{100}{20} = 5 \Rightarrow (5, 10)$

As $y > 0$, then

$y^2 = 20x \Rightarrow y = \sqrt{20x} = \sqrt{20}\sqrt{x} = \sqrt{4}\sqrt{5}\sqrt{x} = 2\sqrt{5}x^{\frac{1}{2}}$

So $y = 2\sqrt{5}x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2\sqrt{5}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \sqrt{5}x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = \frac{\sqrt{5}}{\sqrt{x}}$

At (5, 10), $m_T = \frac{dy}{dx} = \frac{\sqrt{5}}{\sqrt{5}} = 1.$

Gradient of tangent at (5, 10) is $m_T = 1.$

So gradient of normal is $m_N = -1.$

N: $y - 10 = -1(x - 5)$

N: $y - 10 = -x + 5$

N: $x + y - 10 - 5 = 0$

N: $x + y - 15 = 0$

Therefore, the equation of the normal is $x + y - 15 = 0.$

b $xy = 9 \Rightarrow y = 9x^{-1}$

$\frac{dy}{dx} = -9x^{-2} = -\frac{9}{x^2}$

At $x = -\frac{3}{2}$, $m_T = \frac{dy}{dx} = -\frac{9}{(-\frac{3}{2})^2} = -\frac{9}{(\frac{9}{4})} = -\frac{36}{9} = -4$

Gradient of tangent at $(-\frac{3}{2}, -6)$ is $m_T = -4$.

So gradient of normal is $m_N = \frac{-1}{-4} = \frac{1}{4}$.

N: $y + 6 = \frac{1}{4}(x + \frac{3}{2})$

N: $4y + 24 = x + \frac{3}{2}$

N: $8y + 48 = 2x + 3$

N: $0 = 2x - 8y + 3 - 48$

N: $0 = 2x - 8y - 45$

Therefore, the equation of the normal is $2x - 8y - 45 = 0$.

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Quadratic Equations
Exercise D, Question 3

Question:

The point $P(4, 8)$ lies on the parabola with equation $y^2 = 4ax$. Find

a the value of a,

b an equation of the normal to C at P.

The normal to C at P cuts the parabola again at the point Q. Find

c the coordinates of Q,

d the length PQ, giving your answer as a simplified surd.

Solution:

a Substituting $x = 4$ and $y = 8$ into $y^2 = 4ax$ gives

$(8)^2 = 4(a)(4) \Rightarrow 64 = 16a \Rightarrow a = \frac{64}{16} = 4$

So, $a = 4$.

b When $a = 4$, $y^2 = 4(4)x \Rightarrow y^2 = 16x$.

For $P(4, 8), y > 0$, so

$y^2 = 16x \Rightarrow y = \sqrt{16x} = \sqrt{16}\sqrt{x} = 4\sqrt{x} = 4x^{\frac{1}{2}}$

So $y = 4x^{\frac{1}{2}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 4\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = 2x^{-\frac{1}{2}}$

So, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{\sqrt{x}}$

At $P(4, 8), m_T = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{\sqrt{4}} = \frac{2}{2} = 1.$

Gradient of tangent at $P(4, 8)$ is $m_T = 1$.

So gradient of normal at $P(4, 8)$ is $m_N = -1$.

N: $y - 8 = -1(x - 4)$

N: $y - 8 = -x + 4$

N: $y = -x + 4 + 8$

N: $y = -x + 12$

Therefore, the equation of the normal to C at P is $y = -x + 12$.

c Normal **N:** $y = -x + 12$ **(1)**

Parabola: $y^2 = 16x$ **(2)**

Multiplying **(1)** by 16 gives

$16y = -16x + 192$

Substituting **(2)** into this equation gives

$$\begin{aligned}
&16y = -y^2 + 192 \\
&y^2 + 16y - 192 = 0 \\
&(y + 24)(y - 8) = 0 \\
&y = -24, 8
\end{aligned}$$

At P, it is already known that $y = 8$. So at Q, $y = -24$.

Substituting $y = -24$ into $y^2 = 16x$ gives

$(-24)^2 = 16x \Rightarrow 576 = 16x \Rightarrow x = \frac{576}{16} = 36.$

Hence the coordinates of Q are $(36, -24)$.

d The coordinates of P and Q are (4, 8) and $(36, -24)$.

$$\begin{aligned}
AB &= \sqrt{(36-4)^2 + (-24-8)^2} \quad \text{Use } d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \\
&= \sqrt{32^2 + (-32)^2} \\
&= \sqrt{2(32)^2} \\
&= \sqrt{2}\sqrt{(32)^2} \\
&= 32\sqrt{2}
\end{aligned}$$

Hence the exact length AB is $32\sqrt{2}$.

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Quadratic Equations
Exercise D, Question 4

Question:

The point $A(-2,\ -16)$ lies on the rectangular hyperbola H with equation $xy = 32$.

a Find an equation of the normal to H at A.

The normal to H at A meets H again at the point B.

b Find the coordinates of B.

Solution:

a $xy = 32 \Rightarrow y = 32x^{-1}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -32x^{-2} = -\frac{32}{x^2}$

At $A(-2, -16), m_T = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{32}{2^2} = -\frac{32}{4} = -8$

Gradient of tangent at $A(-2, -16)$ is $m_T = -8$.

So gradient of normal at $A(-2, -16)$ is $m_N = \frac{-1}{-8} = \frac{1}{8}$.

N: $y + 16 = \frac{1}{8}(x + 2)$

N: $8y + 128 = x + 2$

N: $0 = x - 8y + 2 - 128$

N: $0 = x - 8y - 126$

The equation of the normal to H at A is $x - 8y - 126 = 0$.

b Normal **N:** $x - 8y - 126 = 0$ **(1)**

Hyperbola H: $xy = 32$ **(2)**

Rearranging **(2)** gives

$y = \frac{32}{x}$

Substituting this equation into **(1)** gives

$$x - 8\left(\frac{32}{x}\right) - 126 = 0$$
$$x - \left(\frac{256}{x}\right) - 126 = 0$$

Multiplying both sides by x gives

$x^2 - 256 - 126x = 0$
$x^2 - 126x - 256 = 0$
$(x - 128)(x + 2) = 0$
$x = 128, -2$

At A, it is already known that $x = -2$. So at B, $x = 128$.

Substituting $x = 128$ into $y = \frac{32}{x}$ gives

$y = \frac{32}{128} = \frac{1}{4}$.

Hence the coordinates of B are $\left(128, \frac{1}{4}\right)$.

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Quadratic Equations
Exercise D, Question 5

Question:

The points $P(4, 12)$ and $Q(-8, -6)$ lie on the rectangular hyperbola H with equation $xy = 48$.

a Show that an equation of the line PQ is $3x - 2y + 12 = 0$.

The point A lies on H. The normal to H at A is parallel to the chord PQ.

b Find the exact coordinates of the two possible positions of A.

Solution:

a The points P and Q have coordinates $P(4, 12)$ and $Q(-8, -6)$.

Hence gradient of PQ, $m_{PQ} = \frac{-6-12}{-8-4} = \frac{-18}{-12} = \frac{3}{2}$

$$\begin{aligned} \text{Hence, } \quad y - 12 &= \tfrac{3}{2}(x-4) \\ 2y - 24 &= 3(x-4) \\ 2y - 24 &= 3x - 12 \\ 0 &= 3x - 2y - 12 + 24 \\ 0 &= 3x - 2y + 12 \end{aligned}$$

The line PQ has equation $3x - 2y + 12 = 0$.

b From part (a), the gradient of the chord PQ is $\frac{3}{2}$.

The normal to H at A is parallel to the chord PQ, implies that the gradient of the normal to H at A is $\frac{3}{2}$.

It follows that the gradient of the tangent to H at A is

$$m_T = \frac{-1}{m_N} = \frac{-1}{\left(\frac{3}{2}\right)} = -\frac{2}{3}$$

$H : xy = 48 \Rightarrow y = 48x^{-1}$

$$\frac{dy}{dx} = -48x^{-2} = -\frac{48}{x^2}$$

At A, $m_T = \frac{dy}{dx} = -\frac{48}{x^2} = -\frac{2}{3} \Rightarrow \frac{48}{x^2} = \frac{2}{3}$

Hence, $2x^2 = 144 \Rightarrow x^2 = 72 \Rightarrow x = \pm\sqrt{72} \Rightarrow x = \pm 6\sqrt{2}$ Note: $\sqrt{72} = \sqrt{36}\sqrt{2} = 6\sqrt{2}$.

When $x = 6\sqrt{2} \Rightarrow y = \frac{48}{6\sqrt{2}} = \frac{8}{\sqrt{2}} = \frac{8\sqrt{2}}{\sqrt{2}\sqrt{2}} = 4\sqrt{2}$.

When $x = -6\sqrt{2} \Rightarrow y = \frac{48}{-6\sqrt{2}} = \frac{-8}{\sqrt{2}} = \frac{-8\sqrt{2}}{\sqrt{2}\sqrt{2}} = -4\sqrt{2}$.

Hence the possible exact coordinates of A are $(6\sqrt{2}, 4\sqrt{2})$ or $(-6\sqrt{2}, -4\sqrt{2})$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise D, Question 6

Question:

The curve H is defined by the equations $x = \sqrt{3}t,\ y = \frac{\sqrt{3}}{t},\ t \in \mathbb{R},\ t \neq 0$.

The point P lies on H with x-coordinate $2\sqrt{3}$. Find:

a a Cartesian equation for the curve H,

b an equation of the normal to H at P.

The normal to H at P meets H again at the point Q.

c Find the exact coordinates of Q.

Solution:

a $xy = \sqrt{3}t \times \left(\frac{\sqrt{3}}{t}\right)$

$xy = \frac{3t}{t}$

Hence, the Cartesian equation of H is $xy = 3$.

b $xy = 3 \Rightarrow y = 3x^{-1}$

$\frac{dy}{dx} = -3x^{-2} = -\frac{3}{x^2}$

At $x = 2\sqrt{3}$, $m_T = \frac{dy}{dx} = -\frac{3}{(2\sqrt{3})^2} = -\frac{3}{12} = -\frac{1}{4}$

Gradient of tangent at P is $m_T = -\frac{1}{4}$.

So gradient of normal at P is $m_N = \frac{-1}{\left(-\frac{1}{4}\right)} = 4$.

At P, when $x = 2\sqrt{3}, \Rightarrow 2\sqrt{3} = \sqrt{3}t \Rightarrow t = \frac{2\sqrt{3}}{\sqrt{3}} = 2$

When $t = 2, y = \frac{\sqrt{3}}{2} \Rightarrow P\left(2\sqrt{3}, \frac{\sqrt{3}}{2}\right)$.

N: $y - \frac{\sqrt{3}}{2} = 4(x - 2\sqrt{3})$

N: $2y - \sqrt{3} = 8(x - 2\sqrt{3})$

N: $2y - \sqrt{3} = 8x - 16\sqrt{3}$

N: $0 = 8x - 2y - 16\sqrt{3} + \sqrt{3}$

N: $0 = 8x - 2y - 15\sqrt{3}$

The equation of the normal to H at P is $8x - 2y - 15\sqrt{3} = 0$.

c Normal **N:** $8x - 2y - 15\sqrt{3} = 0$ **(1)**

Hyperbola H: $xy = 3$ **(2)**

Rearranging **(2)** gives

$$y = \frac{3}{x}$$

Substituting this equation into **(1)** gives

$$8x - 2\left(\frac{3}{x}\right) - 15\sqrt{3} = 0$$

$$8x - \left(\frac{6}{x}\right) - 15\sqrt{3} = 0$$

Multiplying both sides by x gives

$$\begin{aligned} 8x - \left(\frac{6}{x}\right) - 15\sqrt{3} &= 0 \\ 8x^2 - 6 - 15\sqrt{3}x &= 0 \\ 8x^2 - 15\sqrt{3}x - 6 &= 0 \end{aligned}$$

At P, it is already known that $x = 2\sqrt{3}$, so $(x - 2\sqrt{3})$ is a factor of this quadratic equation. Hence,

$$(x - 2\sqrt{3})(8x + \sqrt{3}) = 0$$

$$x = 2\sqrt{3}\,(\text{at } P) \text{ or } x = -\frac{\sqrt{3}}{8}(\text{at } Q).$$

At P, when $x = -\frac{\sqrt{3}}{8}, \Rightarrow \frac{-\sqrt{3}}{8} = \sqrt{3}t \Rightarrow t = \frac{-\sqrt{3}}{8\sqrt{3}} = -\frac{1}{8}$

When $t = -\frac{1}{8}$, $y = \dfrac{\sqrt{3}}{\left(-\frac{1}{8}\right)} = -8\sqrt{3} \Rightarrow Q\left(-\frac{1}{8}\sqrt{3}, -8\sqrt{3}\right)$.

Hence the coordinates of Q are $\left(-\frac{1}{8}\sqrt{3}, -8\sqrt{3}\right)$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise D, Question 7

Question:

The point $P(4t^2,\ 8t)$ lies on the parabola C with equation $y^2 = 16x$. The point P also lies on the rectangular hyperbola H with equation $xy = 4$.

a Find the value of t, and hence find the coordinates of P.

The normal to H at P meets the x-axis at the point N.

b Find the coordinates of N.

The tangent to C at P meets the x-axis at the point T.

c Find the coordinates of T.

d Hence, find the area of the triangle NPT.

Solution:

a Substituting $x = 4t^2$ and $y = 8t$ into $xy = 4$ gives

$(4t^2)(8t) = 4 \Rightarrow 32t^3 = 4 \Rightarrow t^3 = \frac{4}{32} = \frac{1}{8}$.

So $t = \sqrt[3]{\left(\frac{1}{8}\right)}$.

When $t = \frac{1}{2}$, $x = 4\left(\frac{1}{2}\right)^2 = 1$.

When $t = \frac{1}{2}$, $y = 8\left(\frac{1}{2}\right) = 4$.

Hence the value of t is $\frac{1}{2}$ and P has coordinates (1, 4).

b $xy = 4 \Rightarrow y = 4x^{-1}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -4x^{-2} = -\frac{4}{x^2}$

At $P(1, 4)$, $m_T = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4}{(1)^2} = -\frac{4}{1} = -4$

Gradient of tangent at $P(1, 4)$ is $m_T = -4$.

So gradient of normal at $P(1, 4)$ is $m_N = \frac{-1}{-4} = \frac{1}{4}$.

N: $y - 4 = \frac{1}{4}(x - 1)$

N: $4y - 16 = x - 1$

N: $0 = x - 4y + 15$

N cuts x-axis $\Rightarrow y = 0 \Rightarrow 0 = x + 15 \Rightarrow x = -15$

Therefore, the coordinates of N are $(-15, 0)$.

c For $P(1, 4)$, $y > 0$, so

$y^2 = 16x \Rightarrow y = \sqrt{16x} = \sqrt{16}\sqrt{x} = 4\sqrt{x} = 4\sqrt{x} = 4x^{\frac{1}{2}}$

So $y = 4x^{\frac{1}{2}}$

$\frac{dy}{dx} = 4\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = 2x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = \frac{2}{\sqrt{x}}$

At $P(1, 4)$, $m_T = \frac{dy}{dx} = \frac{2}{\sqrt{1}} = \frac{2}{1} = 2.$

Gradient of tangent at $P(1, 4)$ is $m_T = 2$.

T: $y - 4 = 2(x - 1)$

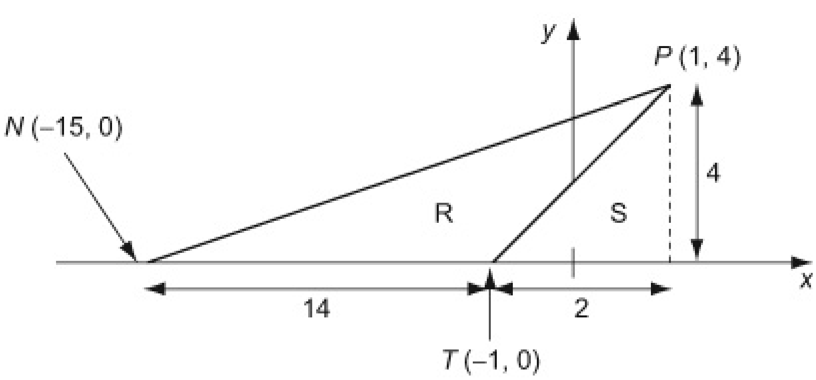
T: $y - 4 = 2x - 2$

T: $0 = 2x - y + 2$

T cuts x-axis $\Rightarrow y = 0 \Rightarrow 0 = 2x + 2 \Rightarrow x = -1$

Therefore, the coordinates of T are $(-1, 0)$.

d



Using sketch drawn, Area Δ NPT
$$\begin{aligned} &= \text{Area}(R + S) - \text{Area}(S) \\ &= \frac{1}{2}(16)(4) - \frac{1}{2}(2)(4) \\ &= 32 - 4 \\ &= 28 \end{aligned}$$

Therefore, Area Δ $NPT = 28$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise E, Question 1

Question:

The point $P(3t^2,\ 6t)$ lies on the parabola C with equation $y^2 = 12x$.

a Show that an equation of the tangent to C at P is $yt = x + 3t^2$.

b Show that an equation of the normal to C at P is $xt + y = 3t^3 + 6t$.

Solution:

a C: $y^2 = 12x \Rightarrow y = \pm\sqrt{12x} = \pm\sqrt{4}\sqrt{3}\sqrt{x} = \pm 2\sqrt{3}x^{\frac{1}{2}}$

So $y = \pm 2\sqrt{3}x^{\frac{1}{2}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm 2\sqrt{3}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \pm\sqrt{3}x^{-\frac{1}{2}}$

So, $\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{\sqrt{3}}{\sqrt{x}}$

At $P(3t^2,\ 6t)$, $m_T = \frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{\sqrt{3}}{\sqrt{3t^2}} = \pm\frac{\sqrt{3}}{\sqrt{3}t} = \frac{1}{t}$.

T: $y - 6t = \frac{1}{t}(x - 3t^2)$

T: $ty - 6t^2 = x - 3t^2$

T: $yt = x - 3t^2 + 6t^2$

T: $yt = x + 3t^2$

The equation of the tangent to C at P is $yt = x + 3t^2$.

b Gradient of tangent at $P(3t^2,\ 6t)$ is $m_T = \frac{1}{t}$.

So gradient of normal at $P(3t^2,\ 6t)$ is $m_N = \frac{-1}{\left(\frac{1}{t}\right)} = -t$.

N: $y - 6t = -t(x - 3t^2)$

N: $y - 6t = -tx + 3t^3$

N: $xt + y = 3t^3 + 6t$.

The equation of the normal to C at P is $xt + y = 3t^3 + 6t$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise E, Question 2

Question:

The point $P\left(6t,\ \frac{6}{t}\right),\ t \neq 0$, lies on the rectangular hyperbola H with equation $xy = 36$.

a Show that an equation of the tangent to H at P is $x + t^2y = 12t$.

b Show that an equation of the normal to H at P is $t^3x - ty = 6(t^4 - 1)$.

Solution:

a H: $xy = 36 \Rightarrow y = 36x^{-1}$

$\frac{dy}{dx} = -36x^{-2} = -\frac{36}{x^2}$

At $P(6t,\ \frac{6}{t})$, $m_T = \frac{dy}{dx} = -\frac{36}{(6t)^2} = -\frac{36}{36t^2} = -\frac{1}{t^2}$

T: $y - \frac{6}{t} = -\frac{1}{t^2}(x - 6t)$ (Now multiply both sides by t^2.)

T: $t^2y - 6t = -(x - 6t)$

T: $t^2y - 6t = -x + 6t$

T: $x + t^2y = 6t + 6t$

T: $x + t^2y = 12t$

The equation of the tangent to H at P is $x + t^2y = 12t$.

b Gradient of tangent at $P(6t,\ \frac{6}{t})$ is $m_T = -\frac{1}{t^2}$.

So gradient of normal at $P\left(6t,\ \frac{6}{t}\right)$ is $m_N = \frac{-1}{\left(-\frac{1}{t^2}\right)} = t^2$.

N: $y - \frac{6}{t} = t^2(x - 6t)$ (Now multiply both sides by t.)

N: $ty - 6 = t^3(x - 6t)$

N: $ty - 6 = t^3x - 6t^4$

N: $6t^4 - 6 = t^3x - ty$

N: $6(t^4 - 1) = t^3x - ty$

The equation of the normal to H at P is $t^3x - ty = 6(t^4 - 1)$.

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Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise E, Question 3

Question:

The point $P(5t^2,\ 10t)$ lies on the parabola C with equation $y^2 = 4ax$, where a is a constant and $t \neq 0$.

a Find the value of a.

b Show that an equation of the tangent to C at P is $yt = x + 5t^2$.

The tangent to C at P cuts the x-axis at the point X and the y-axis at the point Y. The point O is the origin of the coordinate system.

c Find, in terms of t, the area of the triangle OXY.

Solution:

a Substituting $x = 5t^2$ and $y = 10t$ into $y^2 = 4ax$ gives

$$(10t)^2 = 4(a)(5t^2) \Rightarrow 100t^2 = 20t^2 a \Rightarrow a = \frac{100t^2}{20t^2} = 5$$

So, $a = 5$.

b When $a = 5$, $y^2 = 4(5)x \Rightarrow y^2 = 20x$.

$$C\text{: } y^2 = 20x \Rightarrow y = \pm\sqrt{20x} = \pm\sqrt{4}\sqrt{5}\sqrt{x} = \pm 2\sqrt{5}x^{\frac{1}{2}}$$

So $y = \pm 2\sqrt{5}x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \pm 2\sqrt{5}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \pm\sqrt{5}x^{-\frac{1}{2}}$$

So, $\dfrac{dy}{dx} = \pm\dfrac{\sqrt{5}}{\sqrt{x}}$

At $P(5t^2,\ 10t)$, $m_T = \dfrac{dy}{dx} = \dfrac{\sqrt{5}}{\sqrt{5t^2}} = \dfrac{\sqrt{5}}{\sqrt{5}\,t} = \dfrac{1}{t}$.

T: $y - 10t = \dfrac{1}{t}(x - 5t^2)$

T: $ty - 10t^2 = x - 5t^2$

T: $yt = x - 5t^2 + 10t^2$

T: $yt = x + 5t^2$

Therefore, the equation of the tangent to C at P is $yt = x + 5t^2$.

For $(at^2,\ 2at)$ on $y^2 = 4ax$

We *always* get $\frac{d}{dx}(y^2) = 4a$

$2y\frac{dy}{dx} = 4a\frac{dy}{dx} = \frac{2a}{y} = \frac{2a}{2ac} = \frac{1}{t}$

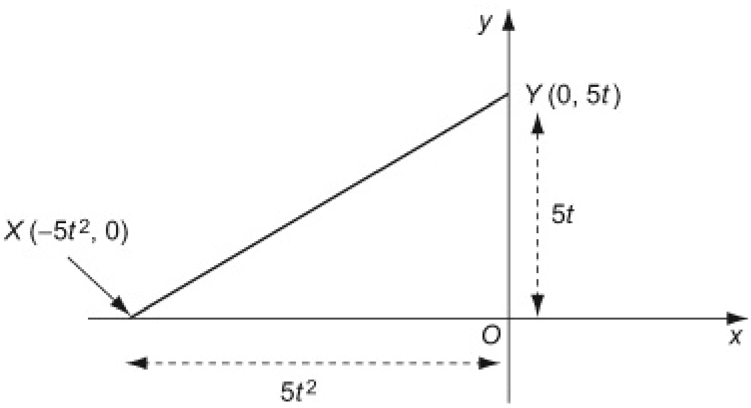
c T: $yt = x + 5t^2$

T cuts x-axis $\Rightarrow y = 0 \Rightarrow 0 = x + 5t^2 \Rightarrow x = -5t^2$

Hence the coordinates of X are $(-5t^2, \ 0)$.

T cuts y-axis $\Rightarrow x = 0 \Rightarrow yt = 5t^2 \Rightarrow y = 5t$

Hence the coordinates of Y are $(0, 5t)$.



Using sketch drawn, Area $\Delta OXY \ = \frac{1}{2}(5t^2)(5t)$
$= \frac{25}{2}t^3$

Therefore, Area $\Delta OXY = \frac{25}{2}t^3$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise E, Question 4

Question:

The point $P(at^2,\ 2at)$, $t \neq 0$, lies on the parabola C with equation $y^2 = 4ax$, where a is a positive constant.

a Show that an equation of the tangent to C at P is $ty = x + at^2$.

The tangent to C at the point A and the tangent to C at the point B meet at the point with coordinates $(-4a,\ 3a)$.

b Find, in terms of a, the coordinates of A and the coordinates of B.

Solution:

a C: $y^2 = 4ax \Rightarrow y = \pm\sqrt{4ax} = \sqrt{4}\sqrt{a}\sqrt{x} = 2\sqrt{a}x^{\frac{1}{2}}$

So $y = 2\sqrt{a}x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2\sqrt{a}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \sqrt{a}x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{x}}$

At $P(at^2,\ 2at)$, $m_T = \frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{at^2}} = \frac{\sqrt{a}}{\sqrt{a}\,t} = \frac{1}{t}$.

T: $y - 2at = \frac{1}{t}(x - at^2)$

T: $ty - 2at^2 = x - at^2$

T: $ty = x - at^2 + 2at^2$

T: $ty = x + at^2$

The equation of the tangent to C at P is $ty = x + at^2$.

b As the tangent **T** goes through $(-4a,\ 3a)$, then substitute $x = -4a$ and $y = 3a$ into **T**.

$t(3a) = -4a + at^2$
$0 = at^2 - 3at - 4a$
$t^2 - 3t - 4 = 0$
$(t+1)(t-4) = 0$
$t = -1,\ 4$

When $t = -1$, $x = a(-1)^2 = a$, $y = 2a(-1) = -2a \Rightarrow (a,\ -2a)$.

When $t = 4$, $x = a(4)^2 = 16a$, $y = 2a(4) = 8a \Rightarrow (16a,\ 8a)$.

The coordinates of A and B are $(a,\ -2a)$ and $(16a,\ 8a)$.

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Quadratic Equations
Exercise E, Question 5

Question:

The point $P\left(4t,\ \frac{4}{t}\right)$, $t \neq 0$, lies on the rectangular hyperbola H with equation $xy = 16$.

a Show that an equation of the tangent to C at P is $x + t^2 y = 8t$.

The tangent to H at the point A and the tangent to H at the point B meet at the point X with y-coordinate 5. X lies on the directrix of the parabola C with equation $y^2 = 16x$.

b Write down the coordinates of X.

c Find the coordinates of A and B.

d Deduce the equations of the tangents to H which pass through X. Give your answers in the form $ax + by + c = 0$, where a, b and c are integers.

Solution:

a H: $xy = 16 \Rightarrow y = 16x^{-1}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -16x^{-2} = -\frac{16}{x^2}$

At $P(4t,\ \frac{4}{t})$, $m_T = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{16}{(4t)^2} = -\frac{16}{16t^2} = -\frac{1}{t^2}$

T: $y - \frac{4}{t} = -\frac{1}{t^2}(x - 4t)$ (Now multiply both sides by t^2.)

T: $t^2 y - 4t = -(x - 4t)$

T: $t^2 y - 4t = -x + 4t$

T: $x + t^2 y = 4t + 4t$

T: $x + t^2 y = 8t$

The equation of the tangent to H at P is $x + t^2 y = 8t$.

b $y^2 = 16x$. So $4a = 16$, gives $a = \frac{16}{4} = 4$.

So the directrix has equation $x + 4 = 0$ or $x = -4$.

Therefore at X, $x = -4$ and as stated $y = 5$.

The coordinates of X are $(-4,\ 5)$.

c T: $x + t^2 y = 8t$

As the tangent **T** goes through $(-4,\ 5)$, then substitute $x = -4$ and $y = 5$ into **T**.

$(-4) + t^2(5) = 8t$
$5t^2 - 4 = 8t$
$5t^2 - 8t - 4 = 0$
$(t-2)(5t+2) = 0$
$t = 2, \ -\frac{2}{5}$

When $t = 2$, $x = 4(2) = 8$, $y = \frac{4}{2} = 2 \Rightarrow (8, \ 2)$.

When $t = -\frac{2}{5}$, $x = 4(-\frac{2}{5}) = -\frac{8}{5}$, $y = \frac{4}{\left(-\frac{2}{5}\right)} = -10 \Rightarrow (-\frac{8}{5}, \ -10)$.

The coordinates of A and B are (8, 2) and $(-\frac{8}{5}, \ -10)$.

d Substitute $t = 2$ and $t = -\frac{2}{5}$ into **T** to find the equations of the tangents to H that go through the point X.

When $t = 2$, **T:** $x + 4y = 16 \Rightarrow x + 4y - 16 = 0$

When $t = -\frac{2}{5}$, **T:** $x + \left(-\frac{2}{5}\right)^2 y = 8\left(-\frac{2}{5}\right)$

T: $x + \frac{4}{25}y = -\frac{16}{5}$

T: $25x + 4y = -80$

T: $25x + 4y + 80 = 0$

Hence the equations of the tangents are $x + 4y - 16 = 0$ and $25x + 4y + 80 = 0$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise E, Question 6

Question:

The point $P(at^2,\ 2at)$ lies on the parabola C with equation $y^2 = 4ax$, where a is a constant and $t \neq 0$. The tangent to C at P cuts the x-axis at the point A.

a Find, in terms of a and t, the coordinates of A.

The normal to C at P cuts the x-axis at the point B.

b Find, in terms of a and t, the coordinates of B.

c Hence find, in terms of a and t, the area of the triangle APB.

Solution:

a C: $y^2 = 4ax \Rightarrow y = \pm\sqrt{4ax} = \sqrt{4}\sqrt{a}\sqrt{x} = 2\sqrt{a}\,x^{\frac{1}{2}}$

So $y = 2\sqrt{a}\,x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2\sqrt{a}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \sqrt{a}\,x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{x}}$

At $P(at^2,\ 2at)$, $m_T = \frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{at^2}} = \frac{\sqrt{a}}{\sqrt{a}\,t} = \frac{1}{t}$.

T: $y - 2at = \frac{1}{t}(x - at^2)$

T: $ty - 2at^2 = x - at^2$

T: $ty = x - at^2 + 2at^2$

T: $ty = x + at^2$

T cuts x-axis $\Rightarrow y = 0$. So,

$0 = x + at^2 \Rightarrow x = -at^2$

The coordinates of A are $(-at^2,\ 0)$.

b Gradient of tangent at $P\left(at^2,\ 2at\right)$ is $m_T = \frac{1}{t}$.

So gradient of normal at $P\left(at^2,\ 2at\right)$ is $m_N = \frac{-1}{\left(\frac{1}{t}\right)} = -t$.

N: $y - 2at = -t(x - at^2)$

N: $y - 2at = -tx + at^3$

N cuts x-axis $\Rightarrow y = 0$. So,

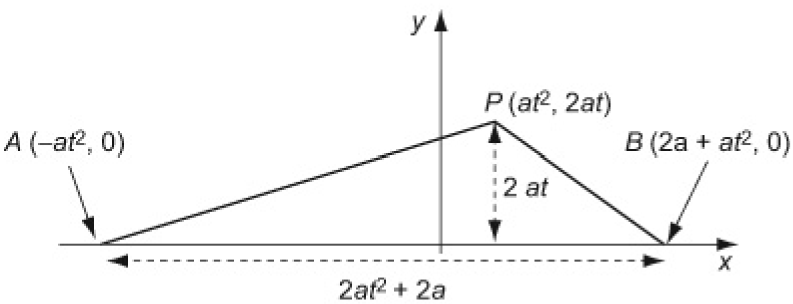
$0 - 2at = -tx + at^3$

$tx = 2at + at^3$

$x = 2a + at^2$

The coordinates of B are $(2a + at^2,\ \ 0)$.

c



Using sketch drawn, Area ΔAPB $= \frac{1}{2}(2a + 2at^2)(2at)$

$= at(2a + 2at^2)$

$= 2a^2t(1 + t^2)$

Therefore, Area $\Delta APB = 2a^2t(1 + t^2)$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise E, Question 7

Question:

The point $P(2t^2,\ 4t)$ lies on the parabola C with equation $y^2 = 8x$.

a Show that an equation of the normal to C at P is $xt + y = 2t^3 + 4t$.

The normals to C at the points R, S and T meet at the point (12, 0).

b Find the coordinates of R, S and T.

c Deduce the equations of the normals to C which all pass through the point (12, 0).

Solution:

a C: $y^2 = 8x \Rightarrow y = \pm\sqrt{8x} = \sqrt{4}\sqrt{2}\sqrt{x} = 2\sqrt{2}x^{\frac{1}{2}}$

So $y = 2\sqrt{2}x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2\sqrt{2}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \sqrt{2}x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = \frac{\sqrt{2}}{\sqrt{x}}$

At $P(2t^2,\ 4t)$, $m_T = \frac{dy}{dx} = \frac{\sqrt{2}}{\sqrt{2t^2}} = \frac{\sqrt{2}}{\sqrt{2}\,t} = \frac{1}{t}$.

Gradient of tangent at $P(2t^2,\ 4t)$ is $m_T = \frac{1}{t}$.

So gradient of normal at $P(2t^2,\ 4t)$ is $m_N = \frac{-1}{\left(\frac{1}{t}\right)} = -t$.

N: $y - 4t = -t(x - 2t^2)$

N: $y - 4t = -tx + 2t^3$

N: $xt + y = 2t^3 + 4t$.

The equation of the normal to C at P is $xt + y = 2t^3 + 4t$.

b As the normals go through (12, 0), then substitute $x = 12$ and $y = 0$ into **N**.

$(12)t + 0 = 2t^3 + 4t$
$12t = 2t^3 + 4t$
$0 = 2t^3 + 4t - 12t$
$0 = 2t^3 - 8t$
$t^3 - 4t = 0$
$t(t^2 - 4) = 0$
$t(t-2)(t+2) = 0$
$t = 0, \ \ 2, \ \ -2$

When $t = 0$, $x = 2(0)^2 = 0$, $y = 4(0) = 0$ $\Rightarrow (0, \ 0)$.

When $t = 2$, $x = 2(2)^2 = 8$, $y = 4(2) = 8$ $\Rightarrow (8, \ 8)$.

When $t = -2$, $x = 2(-2)^2 = 8$, $y = 4(-2) = -8$ $\Rightarrow (8, \ -8)$.

The coordinates of R, S and T are (0, 0), (8, 8) and $(8, \ -8)$.

c Substitute $t = 0, \ 2, \ -2$ into $xt + y = 2t^3 + 4t$. to find the equations of the normals to H that go through the point (12, 0).

When $t = 0$, **N:** $0 + y = 0 + 0. \Rightarrow y = 0$

When $t = 2$, **N:** $x(2) + y = 2(8) + 4(2)$

N: $2x + y = 24$

N: $2x + y - 24 = 0$

When $t = -2$, **N:** $x(-2) + y = 2(-8) + 4(-2)$

N: $-2x + y = -24$

N: $2x - y - 24 = 0$

Hence the equations of the normals are $y = 0$, $2x + y - 24 = 0$ and $2x - y - 24 = 0$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise E, Question 8

Question:

The point $P(at^2,\ 2at)$ lies on the parabola C with equation $y^2 = 4ax$, where a is a positive constant and $t \neq 0$. The tangent to C at P meets the y-axis at Q.

a Find in terms of a and t, the coordinates of Q.

The point S is the focus of the parabola.

b State the coordinates of S.

c Show that PQ is perpendicular to SQ.

Solution:

a C: $y^2 = 4ax \Rightarrow y = \sqrt{4ax} = \sqrt{4}\sqrt{a}\sqrt{x} = 2\sqrt{a}x^{\frac{1}{2}}$

So $y = 2\sqrt{a}x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2\sqrt{a}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \sqrt{a}x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{x}}$

At $P(at^2,\ 2at)$, $m_T = \frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{at^2}} = \frac{\sqrt{a}}{\sqrt{a}t} = \frac{1}{t}$.

T: $y - 2at = \frac{1}{t}(x - at^2)$

T: $ty - 2at^2 = x - at^2$

T: $ty = x - at^2 + 2at^2$

T: $ty = x + at^2$

T meets $y - axis \Rightarrow x = 0$. So,

$ty = 0 + at^2 \Rightarrow y = \frac{at^2}{t} \Rightarrow y = at$

The coordinates of Q are $(0, at)$.

b The focus of a parabola with equation $y^2 = 4ax$ has coordinates $(a, 0)$.

So, the coordinates of S are $(a, 0)$.

c $P(at^2,\ 2at)$, $Q(0, at)$ and $S(a, 0)$.

$$m_{PQ} = \frac{at - 2at}{0 - at^2} = \frac{-at}{-at^2} = \frac{1}{t}.$$

$$m_{SQ} = \frac{0 - at}{a - 0} = -\frac{at}{a} = -t.$$

Therefore, $m_{PQ} \times m_{SQ} = \frac{1}{t} \times -t = -1.$

So PQ is perpendicular to SQ.

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Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise E, Question 9

Question:

The point $P(6t^2,\ 12t)$ lies on the parabola C with equation $y^2 = 24x$.

a Show that an equation of the tangent to the parabola at P is $ty = x + 6t^2$.

The point X has y-coordinate 9 and lies on the directrix of C.

b State the x-coordinate of X.

The tangent at the point B on C goes through point X.

c Find the possible coordinates of B.

Solution:

a C: $y^2 = 24x \Rightarrow y = \pm\sqrt{24x} = \sqrt{4}\sqrt{6}\sqrt{x} = 2\sqrt{6}x^{\frac{1}{2}}$

So $y = 2\sqrt{6}x^{\frac{1}{2}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{6}\left(\frac{1}{2}\right)x^{\frac{1}{2}} = \sqrt{6}x^{-\frac{1}{2}}$

So, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{6}}{\sqrt{x}}$

At $P(6t^2,\ 12t)$, $m_T = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{6}}{\sqrt{6t^2}} = \frac{\sqrt{6}}{\sqrt{6}t} = \frac{1}{t}$.

T: $y - 12t = \frac{1}{t}(x - 6t^2)$

T: $ty - 12t^2 = x - 6t^2$

T: $ty = x - 6t^2 + 12t^2$

T: $ty = x + 6t^2$

The equation of the tangent to C at P is $ty = x + 6t^2$.

b $y^2 = 24x$. So $4a = 24$, gives $a = \frac{24}{4} = 6$.

So the directrix has equation $x + 6 = 0$ or $x = -6$.

Therefore at X, $x = -6$.

c T: $ty = x + 6t^2$ and the coordinates of X are $(-6,\ 9)$.

As the tangent **T** goes through $(-6,\ 9)$, then substitute $x = -6$ and $y = 9$ into **T**.

$t(9) = -6 + 6t^2$
$0 = 6t^2 - 9t - 6$
$2t^2 - 3t - 2 = 0$
$(t-2)(2t+1) = 0$
$t = 2,\ -\frac{1}{2}$

When $t = 2,\quad x = 6(2)^2 = 24,\quad y = 12(2) = 24 \Rightarrow (24,\ 24).$

When $t = -\frac{1}{2},\ x = 6\left(-\frac{1}{2}\right)^2 = \frac{3}{2},\quad y = 12\left(-\frac{1}{2}\right) = -6 \Rightarrow \left(\frac{3}{2},\ -6\right).$

The possible coordinates of B are (24, 24) and $\left(\frac{3}{2},\ -6\right)$.

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise F, Question 1

Question:

A parabola C has equation $y^2 = 12x$. The point S is the focus of C.

a Find the coordinates of S.

The line l with equation $y = 3x$ intersects C at the point P where $y > 0$.

b Find the coordinates of P.

c Find the area of the triangle OPS, where O is the origin.

Solution:

a $y^2 = 12x$. So $4a = 12$, gives $a = \frac{12}{4} = 3$.

So the focus S, has coordinates (3, 0).

b Line l: $y = 3x$ **(1)**

Parabola C: $y^2 = 12x$ **(2)**

Substituting **(1)** into **(2)** gives

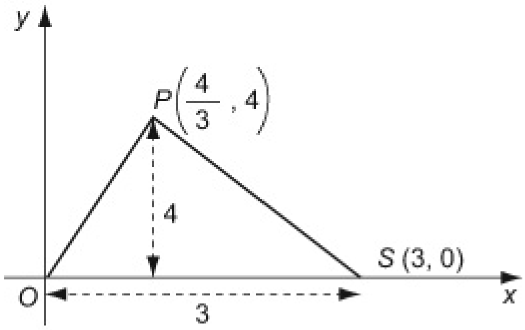
$$(3x)^2 = 12x$$
$$9x^2 = 12x$$
$$9x^2 - 12x = 0$$
$$3x(3x - 4) = 0$$
$$x = 0, \frac{4}{3}$$

Substituting these values of x back into equation **(1)** gives

$$x = 0, y = 3(0) = 0 \Rightarrow (0, 0)$$
$$x = \frac{4}{3}, y = 3\left(\frac{4}{3}\right) = 4 \Rightarrow \left(\frac{4}{3}, 4\right)$$

As $y > 0$, the coordinates of P are $\left(\frac{4}{3}, 4\right)$.

c



Using sketch drawn, Area $\Delta\, OPS = \frac{1}{2}(3)(4)$

$= \frac{1}{2}(12)$

$= 6$

Therefore, Area $\Delta\, OPS = 6$

Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Quadratic Equations
Exercise F, Question 2

Question:

A parabola C has equation $y^2 = 24x$. The point P with coordinates $(k, 6)$, where k is a constant lies on C.

a Find the value of k.

The point S is the focus of C.

b Find the coordinates of S.

The line l passes through S and P and intersects the directrix of C at the point D.

c Show that an equation for l is $4x + 3y - 24 = 0$.

d Find the area of the triangle OPD, where O is the origin.

Solution:

a $(k, 6)$ lies on $y^2 = 24x$ gives

$6^2 = 24k \Rightarrow 36 = 24k \Rightarrow \frac{36}{24} = k \Rightarrow k = \frac{3}{2}$.

b $y^2 = 24x$. So $4a = 24$, gives $a = \frac{24}{4} = 6$.

So the focus S, has coordinates $(6, 0)$.

c The point P and S have coordinates $P\left(\frac{3}{2}, 6\right)$ and $S(6, 0)$.

$$m_l = m_{PS} = \frac{0-6}{6-\frac{3}{2}} = \frac{-6}{\frac{9}{2}} = -\frac{12}{9} = -\frac{4}{3}$$

l: $y - 0 = -\frac{4}{3}(x - 6)$

l: $3y = -4(x - 6)$

l: $3y = -4x + 24$

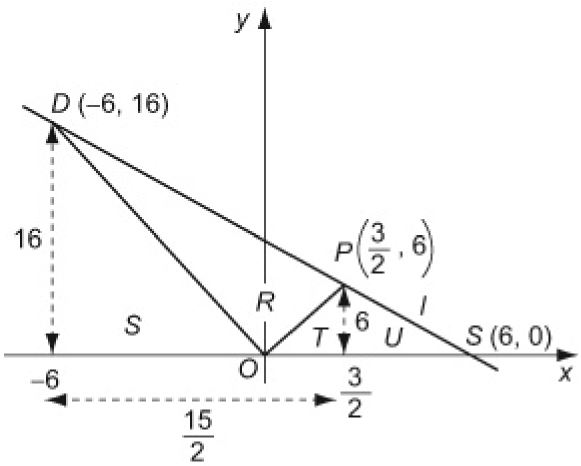
l: $4x + 3y - 24 = 0$

Therefore an equation for l is $4x + 3y - 24 = 0$.

d From (b), as $a = 6$, an equation of the directrix is $x + 6 = 0$ or $x = -6$. Substituting $x = -6$ into l gives:

$4(-6) + 3y - 24 = 0$
$3y = 24 + 24$
$3y = 48$
$y = 16$

Hence the coordinates of D are $(-6, 16)$.



Using the sketch and the regions as labeled you can find the area required. Let Area $\Delta\,OPD = \text{Area}(R)$

Method 1

$$\begin{aligned}
\text{Area}(R) &= \text{Area}(RST) - \text{Area}(S) - \text{Area}(T) \\
&= \tfrac{1}{2}(16+6)\left(\tfrac{15}{2}\right) - \tfrac{1}{2}(6)(16) - \tfrac{1}{2}\left(\tfrac{3}{2}\right)(6) \\
&= \tfrac{1}{2}(22)\left(\tfrac{15}{2}\right) - (3)(16) - \left(\tfrac{3}{2}\right)(3) \\
&= \left(\tfrac{165}{2}\right) - 48 - \left(\tfrac{9}{2}\right) \\
&= 30
\end{aligned}$$

Therefore, Area $\Delta\,OPD = 30$

Method 2

$$\begin{aligned}
\text{Area}(R) &= \text{Area}(RSTU) - \text{Area}(S) - \text{Area}(TU) \\
&= \tfrac{1}{2}(12)(16) - \tfrac{1}{2}(6)(16) - \tfrac{1}{2}(6)(6) \\
&= 96 - 48 - 18 \\
&= 30
\end{aligned}$$

Therefore, Area $\Delta\,OPD = 30$

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Quadratic Equations
Exercise F, Question 3

Question:

The parabola C has parametric equations $x = 12t^2$, $y = 24t$. The focus to C is at the point S.

a Find a Cartesian equation of C.

The point P lies on C where $y > 0$. P is 28 units from S.

b Find an equation of the directrix of C.

c Find the exact coordinates of the point P.

d Find the area of the triangle OSP, giving your answer in the form $k\sqrt{3}$, where k is an integer.

Solution:

a $y = 24t$

So $t = \frac{y}{24}$ **(1)**

$x = 12t^2$ **(2)**

Substitute **(1)** into **(2)**:

$$x = 12\left(\frac{y}{24}\right)^2$$

So $x = \frac{12y^2}{576}$ simplifies to $x = \frac{y^2}{48}$

Hence, the Cartesian equation of C is $y^2 = 48x$.

b $y^2 = 48x$. So $4a = 48$, gives $a = \frac{48}{4} = 12$.

Therefore an equation of the directrix of C is $x + 12 = 0$ or $x = -12$.

c

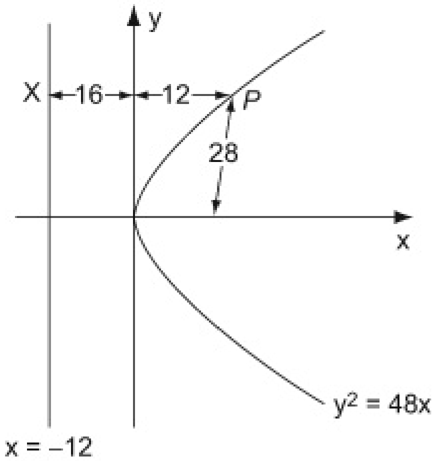
From (b), as $a = 12$, the coordinates of S, the focus to C are $(12, 0)$. Hence, drawing a sketch gives,

The (shortest) distance of P to the line $x = -16$ is the distance XP.

The distance $SP = 28$.

The focus-directrix property implies that $SP = XP = 28$.

The directrix has equation $x = -12$.



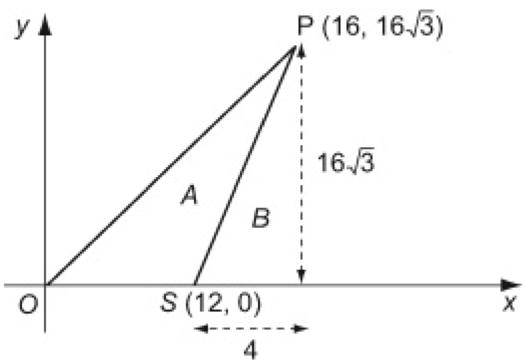
Therefore the x-coordinate of P is $x = 28 - 12 = 16$.

When $x = 16, y^2 = 48(16) \Rightarrow y^2 = 3(16)^2$

As $y > 0$, then $y = \sqrt{3(16)^2} = 16\sqrt{3}$.

Hence the exact coordinates of P are $(16, 16\sqrt{3})$.

d



Using the sketch and the regions as labeled you can find the area required. Let Area $\Delta\, OSP$ = Area(A)

$$\begin{aligned}
\text{Area}(A) &= \text{Area}(AB) - \text{Area}(B)\\
&= \tfrac{1}{2}(16)(16\sqrt{3}) - \tfrac{1}{2}(4)(16\sqrt{3})\\
&= 128\sqrt{3} - 32\sqrt{3}\\
&= 96\sqrt{3}
\end{aligned}$$

Therefore, Area $\Delta\, OSP = 96\sqrt{3}$ and $k = 96$.

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Quadratic Equations
Exercise F, Question 4

Question:

The point $(4t^2,\ 8t)$ lies on the parabola C with equation $y^2 = 16x$. The line l with equation $4x - 9y + 32 = 0$ intersects the curve at the points P and Q.

a Find the coordinates of P and Q.

b Show that an equation of the normal to C at $(4t^2,\ 8t)$ is $xt + y = 4t^3 + 8t$.

c Hence, find an equation of the normal to C at P and an equation of the normal to C at Q.

The normal to C at P and the normal to C at Q meet at the point R.

d Find the coordinates of R and show that R lies on C.

e Find the distance OR, giving your answer in the form $k\sqrt{97}$, where k is an integer.

Solution:

a ***Method 1***

Line: $4x - 9y + 32 = 0$ **(1)**

Parabola C: $y^2 = 16x$ **(2)**

Multiplying **(1)** by 4 gives

$16x - 36y + 128 = 0$ **(3)**

Substituting **(2)** into **(3)** gives

$$y^2 - 36y + 128 = 0$$
$$(y-4)(y-32) = 0$$
$$y = 4, 32$$

When $y = 4$, $4^2 = 16x \Rightarrow x = \frac{16}{16} = 1 \Rightarrow (1, 4).$

When $y = 32$, $32^2 = 16x \Rightarrow x = \frac{1024}{16} = 64 \Rightarrow (64, 32).$

The coordinates of P and Q are (1, 4) and (64, 32).

Method 2

Line: $4x - 9y + 32 = 0$ **(1)**

Parabola C: $x = 4t^2, y = 8t$ **(2)**

Substituting **(2)** into **(1)** gives

$4(4t^2) - 9(8t) + 32 = 0$
$16t^2 - 72t + 32 = 0$
$2t^2 - 9t + 4 = 0$
$(2t - 1)(t - 4) = 0$
$t = \frac{1}{2}, 4$

When $t = \frac{1}{2}$, $x = 4\left(\frac{1}{2}\right)^2 = 1$, $y = 8\left(\frac{1}{2}\right) = 4 \Rightarrow (1, 4)$.

When $t = 4$, $x = 4(4)^2 = 64$, $y = 8(4) = 32 \Rightarrow (64, 32)$.

The coordinates of P and Q are (1, 4) and (64, 32).

b C: $y^2 = 16x \Rightarrow y = \sqrt{16x} = \sqrt{16}\sqrt{x} = 4x^{\frac{1}{2}}$

So $y = 4x^{\frac{1}{2}}$

$\frac{dy}{dx} = 4\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = 2x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = \frac{2}{\sqrt{x}}$

At $(4t^2, 8t)$, $m_T = \frac{dy}{dx} = \frac{2}{\sqrt{4t^2}} = \frac{2}{2t} = \frac{1}{t}$.

Gradient of tangent at $(4t^2, 8t)$ is $m_T = \frac{1}{t}$.

So gradient of normal at $(4t^2, 8t)$ is $m_N = \frac{-1}{\left(\frac{1}{t}\right)} = -t$.

N: $y - 8t = -t(x - 4t^2)$

N: $y - 8t = -tx + 4t^3$

N: $xt + y = 4t^3 + 8t$.

The equation of the normal to C at $(4t^2, 8t)$ is $xt + y = 4t^3 + 8t$.

c Without loss of generality, from part (a) P has coordinates (1, 4) when $t = \frac{1}{2}$ and Q has coordinates (64, 32) when $t = 4$.

When $t = \frac{1}{2}$,

N: $x\left(\frac{1}{2}\right) + y = 4\left(\frac{1}{2}\right)^3 + 8\left(\frac{1}{2}\right)$

N: $\frac{1}{2}x + y = \frac{1}{2} + 4$

N: $x + 2y = 1 + 8$

N: $x + 2y - 9 = 0$

When $t = 4$,

N: $x(4)+y=4(4)^3+8(4)$

N: $4x+y=256+32$

N: $4x+y-288=0$

d The normals to C at P and at Q are $x+2y-9=0$ and $4x+y-288=0$

$\mathbf{N_1}$: $x+2y-9=0$ **(1)**

$\mathbf{N_2}$: $4x+y-288=0$ **(2)**

Multiplying **(2)** by 2 gives

$2\times\mathbf{(2)}:$ $8x+2y-576=0$ **(3)**

$\mathbf{(3)-(1)}:$ $7x-567=0$

$$\Rightarrow 7x=567\Rightarrow x=\frac{567}{7}=81$$

$\mathbf{(2)}\Rightarrow$ $y=288-4(81)=288-324=-36$

The coordinates of R are $(81,-36)$.

When $y=-36$, LHS $=y^2=(-36)^2=1296$

When $x=81$, RHS $=16x=16(81)=1296$

As $LHS=RHS$, R lies on C.

e The coordinates of O and R are (0, 0) and (81, –36).

$$\begin{aligned} OR &= \sqrt{(81-0)^2+(-36-0)^2} \\ &= \sqrt{81^2+36^2} \\ &= \sqrt{7857} \\ &= \sqrt{(81)(97)} \\ &= \sqrt{81}\sqrt{97} \\ &= 9\sqrt{97} \end{aligned}$$

Hence the exact distance OR is $9\sqrt{97}$ and $k=9$.

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Quadratic Equations
Exercise F, Question 5

Question:

The point $P(at^2,\ 2at)$ lies on the parabola C with equation $y^2 = 4ax$, where a is a positive constant. The point Q lies on the directrix of C. The point Q also lies on the x-axis.

a State the coordinates of the focus of C and the coordinates of Q.

The tangent to C at P passes through the point Q.

b Find, in terms of a, the two sets of possible coordinates of P.

Solution:

The focus and directrix of a parabola with equation $y^2 = 4ax$, are $(a, 0)$ and $x + a = 0$ respectively.

a Hence the coordinates of the focus of C are $(a, 0)$.

As Q lies on the x-axis then $y = 0$ and so Q has coordinates $(-a, 0)$.

b C: $y^2 = 4ax \Rightarrow y = \sqrt{4ax} = \sqrt{4}\sqrt{a}\sqrt{x} = 2\sqrt{a}x^{\frac{1}{2}}$

So $y = 2\sqrt{a}x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2\sqrt{a}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \sqrt{a}x^{-\frac{1}{2}}$

So, $\frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{x}}$

At $P(at^2, 2at)$, $m_T = \frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{at^2}} = \frac{\sqrt{a}}{\sqrt{a}t} = \frac{1}{t}$.

T: $y - 2at = \frac{1}{t}(x - at^2)$

T: $ty - 2at^2 = x - at^2$

T: $ty = x - at^2 + 2at^2$

T: $ty = x + at^2$

T passes through $(-a, 0)$, so substitute $x = -a,\ y = 0$ in **T**.

$t(0) = -a + at^2 \Rightarrow 0 = -a + at^2 \Rightarrow 0 = -1 + t^2$

So, $t^2 - 1 = 0 \Rightarrow (t - 1)(t + 1) = 0 \Rightarrow t = 1, -1$

When $t = 1$, $\quad x = a(1)^2 = a$, $\quad y = 2a(1) = 2a \quad \Rightarrow (a, 2a)$.

When $t = -1$, $\ x = a(-1)^2 = a$, $\ y = 2a(-1) = -2a \Rightarrow (a, -2a)$.

The possible coordinates of P are $(a, 2a)$ or $(a, -2a)$.

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Quadratic Equations
Exercise F, Question 6

Question:

The point $P\left(ct,\ \frac{c}{t}\right)$, $c > 0$, $t \neq 0$, lies on the rectangular hyperbola H with equation $xy = c^2$.

a Show that the equation of the normal to H at P is $t^3x - ty = c(t^4 - 1)$.

b Hence, find the equation of the normal n to the curve V with the equation $xy = 36$ at the point (12, 3). Give your answer in the form $ax + by = \mathrm{d}$, where a, b and d are integers.

The line n meets V again at the point Q.

c Find the coordinates of Q.

Solution:

a H: $xy = c^2 \Rightarrow y = c^2x^{-1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -c^2x^{-2} = -\frac{c^2}{x^2}$$

At $P\left(ct, \frac{c}{t}\right)$, $m_T = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{(ct)^2} = -\frac{c^2}{c^2t^2} = -\frac{1}{t^2}$

Gradient of tangent at $P\left(ct, \frac{c}{t}\right)$ is $m_T = -\frac{1}{t^2}$.

So gradient of normal at $P\left(ct, \frac{c}{t}\right)$ is $m_N = \frac{-1}{\left(-\frac{1}{t^2}\right)} = t^2$.

N: $y - \frac{c}{t} = t^2(x - ct)$ (Now multiply both sides by t.)

N: $ty - c = t^3(x - ct)$

N: $ty - c = t^3x - ct^4$

N: $ct^4 - c = t^3x - ty$

N: $t^3x - ty = ct^4 - c$

N: $t^3x - ty = c(t^4 - 1)$

The equation of the normal to H at P is $t^3x - ty = c(t^4 - 1)$.

b Comparing $xy = 36$ with $xy = c^2$ gives $c = 6$ and comparing the point (12, 3) with $\left(ct, \frac{c}{t}\right)$ gives

$ct = 12 \Rightarrow (6)t = 12 \Rightarrow t = 2$. Therefore,

n: $(2)^3x - (2)y = 6((2)^4 - 1)$

n: $8x - 2y = 6(15)$

n: $8x - 2y = 90$

n: $4x - y = 45$

An equation for n is $4x - y = 45$.

c Normal n: $4x - y = 45$ **(1)**

Hyperbola V: $xy = 36$ **(2)**

Rearranging **(2)** gives

$$y = \frac{36}{x}$$

Substituting this equation into **(1)** gives

$$4x - \left(\frac{36}{x}\right) = 45$$

Multiplying both sides by x gives

$$4x^2 - 36 = 45x$$
$$4x^2 - 45x - 36 = 0$$
$$(x - 12)(4x + 3) = 0$$
$$x = 12, -\frac{3}{4}$$

It is already known that $x = 12$. So at Q, $x = -\frac{3}{4}$.

Substituting $x = -\frac{3}{4}$ into $y = \frac{36}{x}$ gives

$$y = \frac{36}{\left(-\frac{3}{4}\right)} = -36\left(\frac{4}{3}\right) = -48.$$

Hence the coordinates of Q are $\left(-\frac{3}{4}, -48\right)$.

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Quadratic Equations
Exercise F, Question 7

Question:

A rectangular hyperbola H has equation $xy = 9$. The lines l_1 and l_2 are tangents to H. The gradients of l_1 and l_2 are both $-\frac{1}{4}$. Find the equations of l_1 and l_2.

Solution:

H: $xy = 9 \Rightarrow y = 9x^{-1}$

$\frac{dy}{dx} = -9x^{-2} = -\frac{9}{x^2}$

Gradients of tangent lines l_1 and l_2 are both $-\frac{1}{4}$ implies

$-\frac{9}{x^2} = -\frac{1}{4}$
$\Rightarrow x^2 = 36$
$\Rightarrow x = \pm\sqrt{36}$
$\Rightarrow x = \pm 6$

When $x = 6$, $6y = 9 \quad \Rightarrow y = \frac{9}{6} = \frac{3}{2} \quad \Rightarrow \left(6, \frac{3}{2}\right)$.

When $x = -6$, $-6y = 9 \quad \Rightarrow y = \frac{9}{-6} = -\frac{3}{2} \quad \Rightarrow \left(-6, -\frac{3}{2}\right)$.

At $\left(6, \frac{3}{2}\right)$, $m_T = -\frac{1}{4}$ and

T: $y - \frac{3}{2} = -\frac{1}{4}(x - 6)$

T: $4y - 6 = -1(x - 6)$

T: $4y - 6 = -x + 6$

T: $x + 4y - 12 = 0$

At $\left(-6, -\frac{3}{2}\right)$, $m_T = -\frac{1}{4}$ and

T: $y + \frac{3}{2} = -\frac{1}{4}(x + 6)$

T: $4y + 6 = -1(x + 6)$

T: $4y + 6 = -x - 6$

T: $x + 4y + 12 = 0$

The equations for l_1 and l_2 are $x + 4y - 12 = 0$ and $x + 4y + 12 = 0$.

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Quadratic Equations
Exercise F, Question 8

Question:

The point P lies on the rectangular hyperbola $xy = c^2$, where $c > 0$. The tangent to the rectangular hyperbola at the point $P\left(ct, \frac{c}{t}\right)$, $t > 0$, cuts the x-axis at the point X and cuts the y-axis at the point Y.

a Find, in terms of c and t, the coordinates of X and Y.

b Given that the area of the triangle OXY is 144, find the exact value of c.

Solution:

a H: $xy = c^2 \Rightarrow y = c^2x^{-1}$

$\frac{dy}{dx} = -c^2x^{-2} = -\frac{c^2}{x^2}$

At $P\left(ct, \frac{c}{t}\right)$, $m_T = \frac{dy}{dx} = -\frac{c^2}{(ct)^2} = -\frac{c^2}{c^2t^2} = -\frac{1}{t^2}$

T: $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$ (Now multiply both sides by t^2.)

T: $t^2y - ct = -(x - ct)$

T: $t^2y - ct = -x + ct$

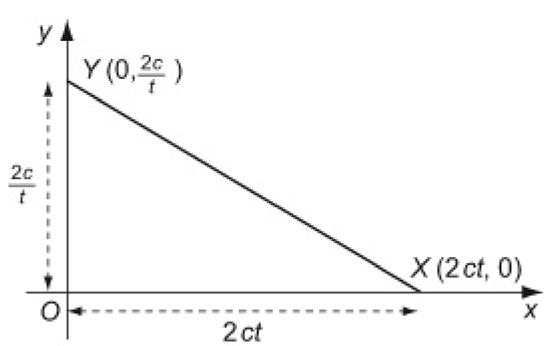
T: $x + t^2y = 2ct$

T cuts x-axis $\Rightarrow y = 0 \Rightarrow x + t^2(0) = 2ct \Rightarrow x = 2ct$

T cuts y-axis $\Rightarrow x = 0 \Rightarrow 0 + t^2y = 2ct \Rightarrow y = \frac{2ct}{t^2} = \frac{2c}{t}$

So the coordinates are $X(2ct, 0)$ and $Y\left(0, \frac{2c}{t}\right)$.

b



Using the sketch, are $\Delta\, OXY = \frac{1}{2}(2ct)\left(\frac{2c}{t}\right) = \frac{4c^2t}{2t} = 2c^2$

As area $\Delta\, OXY = 144$, then $2c^2 = 144 \Rightarrow c^2 = 72$

As $c > 0$, $c = \sqrt{72} = \sqrt{36}\,\sqrt{2} = 6\sqrt{2}$.

Hence the exact value of c is $6\sqrt{2}$.

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Quadratic Equations
Exercise F, Question 9

Question:

The points $P(4at^2,\ 4at)$ and $Q(16at^2,\ 8at)$ lie on the parabola C with equation $y^2 = 4ax$, where a is a positive constant.

a Show that an equation of the tangent to C at P is $2ty = x + 4at^2$.

b Hence, write down the equation of the tangent to C at Q.

The tangent to C at P meets the tangent to C at Q at the point R.

c Find, in terms of a and t, the coordinates of R.

Solution:

a C: $y^2 = 4ax \Rightarrow y = \pm\sqrt{4ax} = \sqrt{4}\sqrt{a}\sqrt{x} = 2\sqrt{a}x^{\frac{1}{2}}$

So $y = 2\sqrt{a}x^{\frac{1}{2}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{a}\left(\frac{1}{2}\right)x^{-\frac{1}{2}} = \sqrt{a}x^{-\frac{1}{2}}$

So, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{a}}{\sqrt{x}}$

At $P(4at^2, 4at)$, $m_T = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{a}}{\sqrt{4at^2}} = \frac{\sqrt{a}}{2\sqrt{a}t} = \frac{1}{2t}$.

T: $y - 4at = \frac{1}{2t}(x - 4at^2)$

T: $2ty - 8at^2 = x - 4at^2$

T: $2ty = x - 4at^2 + 8at^2$

T: $2ty = x + 4at^2$

The equation of the tangent to C at $P(4at^2, 4at)$ is $2ty = x + 4at^2$.

b P has mapped onto Q by replacing t by $2t$, ie. $t \to 2t$

So, $P(4at^2, 4at) \to Q(16at^2, 8at) = Q(4a(2t)^2, 4a(2t))$

At Q, **T** becomes $2(2t)y = x + 4a(2t)^2$

T: $2(2t)y = x + 4a(2t)^2$

T: $4ty = x + 4a(4t^2)$

T: $4ty = x + 16at^2$

The equation of the tangent to C at $Q(16at^2, 8at)$ is $4ty = x + 16at^2$.

c T_P: $2ty = x + 4at^2$ **(1)**

T_Q: $4ty = x + 16at^2$ **(2)**

(2) – (1) gives

$2ty = 12at^2$

Hence, $y = \frac{12at^2}{2t} = 6at$.

Substituting this into **(1)** gives,

$2t(6at) = x + 4at^2$

$12at^2 = x + 4at^2$

$12at^2 - 4at^2 = x$

Hence, $x = 8at^2$.

The coordinates of R are $(8at^2, 6at)$.

Solutionbank FP1
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Quadratic Equations
Exercise F, Question 10

Question:

A rectangular hyperbola H has Cartesian equation $xy = c^2$, $c > 0$. The point $\left(ct,\ \frac{c}{t}\right)$, where $t \neq 0$, $t > 0$ is a general point on H.

a Show that an equation an equation of the tangent to H at $\left(ct,\ \frac{c}{t}\right)$ is $x + t^2y = 2ct$.

The point P lies on H. The tangent to H at P cuts the x-axis at the point X with coordinates $(2a, 0)$, where a is a constant.

b Use the answer to part **a** to show that P has coordinates $\left(a,\ \frac{c^2}{a}\right)$.

The point Q, which lies on H, has x-coordinate $2a$.

c Find the y-coordinate of Q.

d Hence, find the equation of the line OQ, where O is the origin.

The lines OQ and XP meet at point R.

e Find, in terms of a, the x-coordinate of R.

Given that the line OQ is perpendicular to the line XP,

f Show that $c^2 = 2a^2$,

g find, in terms of a, the y-coordinate of R.

Solution:

a H: $xy = c^2 \Rightarrow y = c^2x^{-1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -c^2x^{-2} = -\frac{c^2}{x^2}$$

At $\left(ct, \frac{c}{t}\right)$, $m_T = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{(ct)^2} = -\frac{c^2}{c^2t^2} = -\frac{1}{t^2}$

T: $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$ (Now multiply both sides by t^2.)

T: $t^2y - ct = -(x - ct)$

T: $t^2y - ct = -x + ct$

T: $x + t^2y = 2ct$

An equation of a tangent to H at $\left(ct, \frac{c}{t}\right)$ is $x + t^2y = 2ct$.

b **T** passes through $X(2a, 0)$, so substitute $x = 2a, y = 0$ into **T**.

$$(2a) + t^2(0) = 2ct \Rightarrow 2a = 2ct \Rightarrow \frac{2a}{2c} = t \Rightarrow t = \frac{a}{c}$$

Substitute $t = \frac{a}{c}$ into $\left(ct, \frac{c}{t}\right)$ gives

$$P\left(c\left(\frac{a}{c}\right), \frac{c}{\left(\frac{a}{c}\right)}\right) = P\left(a, \frac{c^2}{a}\right).$$

Hence P has coordinates $P\left(a, \frac{c^2}{a}\right)$.

c Substituting $x = 2a$ into the curve H gives

$$(2a)y = c^2 \Rightarrow y = \frac{c^2}{2a}.$$

The y-coordinate of Q is $y = \frac{c^2}{2a}$.

d The coordinates of O and Q are $(0, 0)$ and $\left(2a, \frac{c^2}{2a}\right)$.

$$m_{OQ} = \frac{\frac{c^2}{2a} - 0}{2a - 0} = \frac{c^2}{2a(2a)} = \frac{c^2}{4a^2}$$

$$OQ: \; y - 0 = \frac{c^2}{4a^2}(x - 0)$$

$$OQ: \; y = \frac{c^2 x}{4a^2}. \quad \textbf{(1)}$$

The equation of OQ is $y = \frac{c^2 x}{4a^2}$.

e The coordinates of X and P are $(2a, 0)$ and $\left(a, \frac{c^2}{a}\right)$.

$$m_{XP} = \frac{\frac{c^2}{a} - 0}{a - 2a} = \frac{\frac{c^2}{a}}{-a} = -\frac{c^2}{a^2}$$

$$XP: \; y - 0 = -\frac{c^2}{a^2}(x - 2a)$$

$$XP: \; y = -\frac{c^2}{a^2}(x - 2a) \quad \textbf{(2)}$$

Substituting **(1)** into **(2)** gives,

$$\frac{c^2 x}{4a^2} = -\frac{c^2}{a^2}(x - 2a)$$

Cancelling $\frac{c^2}{a^2}$ gives,

$\frac{x}{4} = -(x - 2a)$

$\frac{x}{4} = -x + 2a$

$\frac{5x}{4} = 2a$

$x = \frac{4(2a)}{5} = \frac{8a}{5}$

The x-coordinate of R is $\frac{8a}{5}$.

f From earlier parts, $m_{OQ} = \frac{c^2}{4a^2}$ and $m_{XP} = -\frac{c^2}{a^2}$

OP is perpendicular to $XP \Rightarrow m_{OQ} \times m_{XP} = -1$, gives

$m_{OQ} \times m_{XP} = \left(\frac{c^2}{4a^2}\right)\left(-\frac{c^2}{a^2}\right) = \frac{-c^4}{4a^4} = -1$

$-c^4 = -4a^4 \Rightarrow c^4 = 4a^4 \Rightarrow \left(c^2\right)^2 = 4a^4$

$c^2 = \sqrt{4a^4} = \sqrt{4}\sqrt{a^4} = 2a^2$.

Hence, $c^2 = 2a^2$, as required.

g At $R, x = \frac{8a}{5}$. Substituting $x = \frac{8a}{5}$ into $y = \frac{c^2 x}{4a^2}$ gives,

$y = \frac{c^2}{4a^2}\left(\frac{8a}{5}\right) = \frac{8ac^2}{20a^2}$

and using the $c^2 = 2a^2$ gives,

$y = \frac{8a(2a^2)}{20a^2} = \frac{16a^3}{20a^2} = \frac{4a}{5}$.

The y-coordinate of R is $\frac{4a}{5}$.